Neue Gradnetzkombinationen
(New Graticule Combinations)
Oswald Winkel, 1921

Petermanns Mitteilungen, v. 67, Dec., p. 248-252.

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Translator‘s Remarks:

The Mercator-Sanson projection referenced in this article is also known as the Sanson-Flamsteed or Sinusoidal projection.

I used the following translations for (old) expressions that were not mentioned in the dictionary (see figures as well for understanding):

Grenzkreis: bounding circle
Grundkreis: Base circle
Hauptkleinkreis: Main small circle
Pollinie: pole line (def.: representation of the terrestrial pole as a line)

Definitions instead of translations:

Abweitung: Decrease of the length of the parallels on the sphere or spheroid towards the poles
Abweitungstreu (adj.): Preservation of the Principal scale along all the Parallels of Latitude in the Normal Aspect of the projection.
Schnittwinkelschiefe: no rectangular angles present (oblique aspect)

New Graticule Combinations

by Oswald Winkel, Leipzig (6 figures)

Introduction to cylindrical projections

It is a well-known fact that zones of a larger width can only be represented advantageously if the projection is carried through on an area that is cutting the sphere and not an area that is touching the sphere. Under this condition, in the case of a cylindrical design, the result is a rectangular Plate Carrée that has disadvantages. Strictly, these maps should only be developed for zones, that have moderate distances of bounding circles (η_1) with respect to the base circle (*Grundkreis*). The cartographer who often has to reproduce very wide zones is missing advice from mathematicians in these cases. To make an end to this state I have developed three geometrically characteristic designs of which the characteristics are presented as follows:

Special cases: a) The representation of zones of a width > 2 × 20°

Design 1a:

Pseudo-cylindrical figure of a zone of 100° length and 2 × 30° width, $n = \cos\eta_0 = \frac{\cos\eta_1}{\cos^2\eta_1/2}$ [1]

As it is known, in the case of true cylindrical projections the convergence of the great circles is annulled, resulting in a severe change of area. This disadvantage is especially noticeable if the zone that is represented is larger than 2 × 20° great circle degrees. There is no doubt, that convergence can only be achieved by application of pseudo-cylindrical projection. In "Petermanns Mitteilungen" 1913 II, Table 46, I showed a rectangular Plate Carrée of the Atlantic Ocean. The percentual stretching of the bounding circles (*Grenzkreise*) ($\pm\eta_1$), expressed in %, equals the percentual shortening of the base circle (*Grundkreis*) (η).

One has $n = \cos\eta_0 = \frac{\cos\eta_1}{\cos^2\eta_1/2}$ as well as $x = \text{arc}\,\xi\, n$ and $y = \text{arc}\,\eta$.

If one marks the small circle between any point and the central great circle ('mittlerer Hauptkreis'; that is the deviation) with $\text{arc}\,\xi\cos\eta$, the mean $\text{arc}\,\xi\left(\frac{n+\cos\eta}{2}\right) = x$ together with $\text{arc}\,\eta = y$ then yields a pseudo-cylindrical projection, which is very useful for the reproduction of zones that are wider than 2 × 20°. The x and y are the mean of x, y of the Mercator-Sanson projection and a rectangular Plate Carrée. I usually write

$$W = \frac{Pl\left(n = \frac{\cos\eta_1}{\cos^2\eta_1/2}\right) + MS}{2}$$

This principle that can be applied to zones up to 2 × 70° width, is the basis for the dotted graticule on Fig. I, for which $n = \cos\eta_0 = \cos 21°51' = 0.928$. In this figure furthermore the rectangular Plate Carrée $n = \cos 21°51'$ (complete lines) and the outer great circles (dotted lines) that result out of plotting of the '*Abweitungen*', are drawn. Thus the characteristics of the projection can easily be noticed. The result is a projection on an area that "nestles" well to the area of the sphere. Values for the "*relative*" distortions – only valid for the points of the *central* great circle:

$$n = \frac{\cos\eta_1}{\cos^2\eta_1/2} = \cos 21°51' = 0.928..$$

[1] This expression describes the radius of those small circles, in which the sphere with the radius = 1 is cutted by the cylinder projection.

	A	b	S	2ω
Bounding circle $\eta_1 = \pm30°$	1.036	1.000	1.036	2°1′
	(1.072)	(1.000)	(1.072)	(3°58′)
Main small circle $\eta_0 = \pm21°51'$	1.000	1.000	1.000	0°0′
Base circle $\eta = \pm0°$	1.000	0.964	0.964	2°6′
	(1.000)	(0.928)	(0.928)	(4°16′)

(in parenthesis the distortions for the *true* cylindrical form)

Characteristics of the design:

The areas of the stripes, that are limited by the small circles, that in the case of the rectangular Plate Carrée show a huge ratio of enlargement/reduction, are changed by less than 50%, the total areal distortion (the effect is a reduction) is substantially diminished. The small circles are parallel straight lines, with equal distance from each other. This important didactical element is preserved. The convergence appears, whereas the rectangular grid disappears. Instead the "*mittlere Schnittwinkelschiefe*" arises as a result of the mean, not absolute convergence. Exact measurements of length are only easily possible on all small circles, it is difficult on the great circles, except the central one.

The advantages of this image are thus clearly visible so that numerical calculations are almost not necessary to recommend its application.

The construction of the graticule in normal position is very easy: One plots the true lengths for the arc η on the central great circle. Through these grid points the straight small circles (parallels) are drawn. After that one draws the outer great circles for the true secant cylindrical projection and for the Mercator-Sanson projection. Then one determines the line of the pseudo secant cylindrical projection which is located exactly in the middle between the outer great circles of the two projections. The small circles are then divided into equal intervals and the missing great circles can then be drawn. The graticule for the *pseudo* secant cylindrical projection is then complete. Much more complicated, of course, is the development of the non-normal forms. I suggest a presentation of an example that proves the production of excellent images with the before mentioned procedure.

Map of Africa and Europe (Fig. II) in transverse aspect, equidistant[2], pseudo-cylindrical projection

The zone in question has a length of 2 × 50° and a width of 2 × 30°. The relations that are mentioned for the understanding of the theory of Fig. I reappear but the difference is that a transverse aspect position of the projection cylinder has to be assumed. The base circle (*Grundkreis*) is the meridian 20° E of Greenwich. This is the x-axis. On it, the origin for the right-angled coordinates, that is C1, is assumed in 10° N. Here lies the y-axis and one finds the points of zero distortion on the equator in 110° E and 70° W of Greenwich. First, all ξ are determined with the help of Hammer's table of azimuts $q_0 = 0$ through the operation 90°-δ for the case that C1 is located on the equator. But since C1 has to be assumed in 10°N, all ξ-values north of the central great circle have to be reduced by 10°, whereas those in the south have to be increased by the same amount. Values of ξ are determined with the help of Hammer's table showing the spherical distances $\varphi_0 = 0°$ out of 90°-δ. The processes are best made obvious with a practical example.

[2] Equidistant I use to call those projections where the distances of the small circles are projected length true. This is valid for nonnormal projections as well.

Task: Search x, y for the grid point 60°N and 40°E (λ = 20°), (Nomination in minutes based on the seamile).

First one has n = cos η_0 = cos 21°51′ = 0.982.. (log = 0.9676430 – 1)

ξ = 90° - (α + 10°) = 90° -(28°28.9′ + 10°) = 51°31.1′ = 3091.1′

η = 90° - δ = 90° - 80°9.2′ = 9°50.8′ = 590.8′

$$x = \frac{arc\xi\, n + arc\xi \cos\eta}{2}$$

log ξ = 3,4901131

+ log n = 9.9676430 - 10

3.4577651

log ξ = 3.4901131

+ log cos η = 9.9935548 - 10

3.4836679

arc ξ cos n = 2869.2′

arc ξ cos η = 3045.6′

x = 2869.2′

+ 3045.6′

5914.8′/2 = 2957.4′; y = 590.8′

To introduce the projection process easier into practice, I add the

Table of rectangular coordinates x, y for the secant cylindrical projection in mean convergence of Africa and Europe

		λ = 0°	10°	20°	30°	40°
φ = 60°	X	2892′	2909′	2957′	3040′	3158′
	Y	0	299	591	869	1125
50°	X	2314	2331	2384	2475	2608
	Y	0	385	762	1125	1464
40°	X	1735	1752	1804	4894	2029
	Y	0	459	911	1351	1770
30°	X	1157	1172	1219	1300	1425
	Y	0	519	1034	1540	2030
20°	X	578	591	629	695	798
	Y	0	564	1125	1682	2230
10°	X C1	0	9	35	81	151
	Y	0	591	1181	1770	2356
0°	X	578	574	560	537	508
	Y	0	600	1200	1800	2400
-10°	X	1157	1157	1157	1161	1172
	Y	0	591	1181	1770	2356
-20°	X	1735	1740	1754	1782	7833
	Y	0	564	1125	1682	2230
-30°	X	2314	2322	2349	2398	2480
	Y	0	519	1034	1540	2030
-40°	X	2892	2904	2940	3005	3108
	Y	0	459	911	1351	1770

The measurement of lengths and areas.

For the determination of the small circle scales the following remarks are necessary: m is a true length; $\underline{m}$ the enlarged/reduced length which is obtained for any small circle from $\underline{m} = m \times k$, in which k is the map parallel $\underline{\eta}$ /sphere parallel η ratio that is given by $\frac{\cos\eta_0 + \cos\eta}{2} : \cos\eta$.

Example: η is 10°; m = 4000 km, 4000 km in 1:85 000 000 = 47.059 mm

$$\cos\eta_0 = 0.9828203$$
$$+\cos\eta = 0.984810$$
$$\frac{\cos\eta_0 + \cos\eta}{2} = 1.913013 / 2 = 0.956506$$
$$\log \frac{\cos\eta_0 + \cos\eta}{2} = 0.9806877 - 1$$
$$-\log\cos\eta \underline{= 0.9933515 -1}$$
$$\log k = 0.9873362 - 1$$
$$\log m = 1.6726427$$
$$+\log k \underline{= 0.9873362 - 1}$$
$$\log \underline{m} = 1.6599789$$
$$m = 45.71 \text{ mm}$$

In the case of a pseudo secant cylindrical projection, exact measurements of lengths in the direction of the great circles cannot be obtained easily. Nevertheless this desirable aim can be achieved by creating a “kilometer graticule” (see Fig. III). Plans often have a square system to find objects easily. Each side of the square corresponds to a certain (metric) measure so that distance measurements can be obtained sufficiently exact without the use of a scale. This idea is the basis for my “kilometer graticule” but it cannot be carried through easily because of spherical elements.

The construction of the “kilometer graticule” starts with the plot of the arc η = 1000 km, 2000 km etc. on the central great circle, then for the arc η the ang η are determined, then the small circles are drawn. On each of these, the kilometer division following the laws of projection is drawn as well and the lines of the graticule are completed. The transfer of the graticule on transparent paper that is put over the map allows exact measurements of lengths not only in the direction of the small circles that are important in the sense of the projection but also in the direction of those small circles that run perpendicular on the sphere to those mentioned just before.

Design 2a) (no figure presented)

Pseudo cylindrical projection of a zone of 100° length and 2 × 30° width, $n = (1 + \cos\eta_1) / 2$

Design 2a is different from design 1a only in the choice of n. The relative distortions, that shall be sufficient for the presentation of the central great circle are as follows:

$n = \cos\eta_0 = (1 + \cos\eta_1) / 2 = \cos 21°5' = 0.933...$

	A	b	S	2ω
Bounding circle $\eta_1 = \pm 30°$	1.041	1.000	1.041	2°18′
Main small circle $\eta_0 = \pm 21°5'$	1.000	1.000	1.000	0°0′
Base circle $\eta = \pm 0°$	1.000	0.967	0.967	1°57′

The result of this procedure is that distortions near the equator are more favourable at the expense of those near the poles (assuming the sphere’s circle in the middle as great circle). The distortion of the whole area is less than in design 1a! $n = (1 + \cos\eta_1) / 2$, I want to call “golden cut for the sphere”.

Design 3a) (no figure presented)

Pseudo cylindrical projection of a zone of 100° length and 2 × 30° width, $n = \sin \eta_1 / \text{arc}\ \eta_1$

Values for the "relative" distortions, only valid for points of the central great circle n = cos 17°16′ = 0.955...

	A	B	S	2ω
Bounding circle $\eta_1 = \pm 30°$	1.051	1.000	1.051	2°52′
Main small circle $\eta_0 = \pm 17°16'$	1.000	1.000	1.000	0°0′
Base circle $\eta = \pm 0°$	1.000	0.977	0.977	1°18′

The figure's area equals the area of the original not only with regard to the whole area but for the stripes within the great circles, too. The areas near the equator are represented more favourable than in 1a and 2a, of course at a larger expense of those near to the poles. Remarkably, the main small circles are still located outside the zone $\pm \eta_1 / 2$. The question arises which one of the three forms, characterized by n, is the better one. To my mind preference should "generally" be given to design 1a which has the most advantages. In the case of the representation of the "whole" surface of the sphere, "form 2a" is superior because of the smaller reduction of the total area.

b) The representation of the "whole" earth surface.

Designs 1b, 2b (for both see introduction), 3b

In the case of 1b where $\eta_1 = 90°$, the rectangular Plate Carrée vanishes, the reduction of the total area increases too much (absolute reduction of area) because both hemispheres would be represented in one, then achieving an absolute convergence. As a limitation for the use of form 1a, I have already regarded the 2 × 70° zone width. The limit $n = (1 + \cos \eta_1) / 2$ is 2b. n is = cos 60° = 0.5. The equator in the case of the pseudo form makes up ¾, the pole line ¼ of the basis of a tangent cylinder. Form 3b ($\sin \eta_1 / \text{arc}\ \eta_1$) is 3a, extended to the whole surface area (Fig. IV). Shifting its outer great circles in a way that the base circle appears '*abweitungstreu*', the pole line/equator ratio will be of the kind of Eckerts bulge projection V.

Eckerts projection V is therefore a pseudo tangent cylindrical projection!

Because the sphere's surface must be assumed as a 2 × 90° wide zone and wide zones only yield a map more similar to the sphere in the projection, if the projection is carried through on an area that cuts the sphere, it becomes clear that M. Eckerts bulge projections are not so fit for the representation of so called "planispheres" as assured in *Petermanns Mitteilungen 1906, p. 102*.

The notion "bulge projection" (polarogkoid) has to be avoided because it is nothing but "pseudo tangent cylindrical projections".

In order to recognize the true character of projection V the process of my steps were as follows:

R is the radius of the sphere, r the bulge equals R × 0.882026, r / R is 1:1,133754 and R = r × 1.133754.

Assuming η as 30° and scale 1:250 000 000, r = 22.475 mm. One finds for projection V: $x = r + (r \times \cos \eta)$ = 65.8775 mm, that is the length from arc 180° to η = 30°. Multiplying this x with 1.133754 x yields 74.69 mm. This value is obtained as well forming the average out of x of the square Plate Carrée and x of the Mercator-Sanson projection:

x for square Plate Carrée	= 80.05 mm
x for Merc.-San.	= 69.33 mm
	= 149.38 mm / 2 = 74.69 mm.

The result is that Eckerts projection V is situated exactly between the square Plate Carrée and the Mercator-Sanson projection, if the equator and central meridian are drawn length true. But because this would lead to an increase in the whole area, Eckert removes the surplus in reducing the reduction proportion.

This procedure that abandones the "natural" scale ratio is of course correct, because there is no other way for "pseudo tangent cylindrical projections" of this type that yields absolute equivalence.
In this context, when creating maps, the indication of the scale ratio, for which the earth's radius is decisive, is desirable in the future. This is also valid for Mollweides '*abweitungsgleiche'* tangent cylindrical projections.

Eckert's images with the half pole line suffer of course a neglection of the large distortions of polar areas. The result is that the "most appropriate expedient solid" for map projections can not be the half bulge. But because this is also valid for Eckerts projection VI, where only a change in the distance of the parallels takes place to yield the "absolute" equivalence, I can not appreciate this form as a final solution to this problem. For the equivalent pseudo secant cylindrical projection it is best to keep up n = sin η_1 / arc η_1. But then the distances of the small circles have to be installed in a way that those stripes that extend in the direction of the small circles have the same area as the original.

I have to add that when choosing n = $\frac{\cos\eta_1}{\cos^2\eta_1/2}$ and n = (1 + cos η_1) / 2 my projections can be drawn so that total equivalence is kept up. R of the sphere is enlarged to R^ and equidistance is replaced by equality of distances. The form of the graticule images do not suffer any change but the calculation is getting more difficult. Despite the larger reduction of the total area, especially in the case of equidistant planispheres I want to recommend to hold to the equidistance because of the associated length- truth in the central great circle and the two small circles. The combination of the rectangular Plate Carrée with the equidistant tangent cylindrical projection à la Mollweide = W1 leads to $\frac{Pl(n<1)+W_1}{2}$ (see Fig. V). The '*Schiefschnittigkeit'* is improved but the polar parts are distorted too much. In the case W1 for the fundamental circle one has ® = arc 90°, η gets $\underline{\eta}$ and x = arc ξ × cos η × v, with v = cos $\underline{\eta}$/ cos η.

Thus we have for $\frac{Pl(n<1)+W_1}{2}$ x = $\frac{arc\xi(n+\cos\eta\cdot v)}{2}$ and y = arc η.

More important still than $\frac{Pl(n<1)+MS}{2}$ and $\frac{Pl(n<1)+W_1}{2}$ is the combination of the true Plate Carrée with Aitoff's equidistant planisphere, that is $\frac{Pl(n<1)+A}{2}$.

For Aitoff's azimuthal projection I found x = 2 arc δ × sin α from Δλ/2, y = arc δ × cos α from Δλ / 2, that is: δ and α have to be calculated with λ / 2, if x, y shall be valid for the length λ. Thus for the "Tripel projection"

$$\frac{Pl(n<1)+A}{2}$$

$$x = \frac{arc\xi\, n + 2\, arc\delta\cdot\sin\alpha\ von\ \Delta\lambda/2}{2}$$

$$y = \frac{arc\eta + arc\delta\cdot\cos\alpha\ \ von\,\Delta\lambda/2}{2}$$

This last projection (Fig. VI) is the most important, it is almost total equivalent, has an improved '*Schiefschnittigkeit'* compared to $\frac{Pl(n<1)+MS}{2}$, leads back the polar distortions of $\frac{Pl(n<1)+W_1}{2}$ to the right proportion, thus eliminating the disadvantage of its predecessors.

A short remark to the relative distortions: I acknowledge the very friendly help of Prof. Dr. Arthur Krause (Leipzig) thankfully. In the figures the points without distortions are marked with an x.

Summary of the main results

1. For the first time *pseudo* secant cylindrical projections are develloped and reasons for the choice of n are given.
2. Observing the geometrical characteristics of n, for equidistant cylindrical projections the determination of those widths was possible where the change in n can be possible without danger for the distortions. A wrong choice of the important constant n can be easily avoided in the future.
3. A “kilometer graticule” is suggested in order to enable measurements of lengths and perhaps areas.
4. It is shown that *pseudo* secant cylindrical projections will displace true ones if the width of the zone exceeds 2 × 20°. In this case the convergence of the great circles has to be considered.
5. The rectangular character of the graticule has to be abandoned and it appears in the case of “secant cylindrical projections” as well the “mean convergence” as a new factor for the useable design of the graticule. Appearing as a “mean total area distortion” thus reducing the absolute distortions that appear in the case of “true” forms, by 50%.
6. If the absolute convergence has not to be maintained, that is when representing zones of less than 2 × 90° width, the more true representation of the sphere’s surface succeeds through the combination of the rectangular Plate Carrée (carefully chosen n) with Aitoff‘s projection.

   Prof. Wagner writes in the 4th edition of his Standard Manual of Geography (I, p.197): “In the production of an almost true image of the earth’s surface and its parts culminates the whole mathematical geography”. As was pointed out already in the introduction, the task to represent very wide zones as less distorted as possible is very important because of its practical significance. It was unsolved until today, remarkably. In the case that the form of such a zone requires the cylindrical design, there is a numerical solution available now. It was found in the arithmetic average of the x, y that can be formed, if well determined, true secant cylindrical projections are combined with the planispheres.

   It is obvious to use these principles that I demonstrated in the case of cylindrical designs for such spherical zones as well, for which first of all conical projections have to stand up for. The requirements to fulfill this problem are now given so that the most true representation of any part of the earth’s surface will be possible, thus filling a gap in the science of cartography. In the framework of methodology, my projections have to be considered as a new category of designs.

   Prof. M. Eckert and Hofrat M. Nell both were on the right track in search of useable projections.

   The art of graticule projection is to spread the inevitable distortions over the map in a way that a map image develops that resembles the sphere`s image. My clues are appropriate tools in this context.

Verzerrungen zu Eckerts V = (Pl(n = 1) + MS)/2 (graphisch nicht dargestellt) s. Pet. Mitt. 1906 Taf. 8

	λ = 0°				λ = 90°				λ = 180°			
	a	b	s	2ω	a	b	s	2ω	a	b	s	2ω
φ = 0°	1.000	1.000	1.000	0° 0'	1.000	1.000	1.000	0° 0'	1.000	1.000	1.000	0° 0'
30°	1.077	1.000	1.077	4°16'	1.257	0.857	1.077	21°50'	1.505	0.716	1.077	41°38'
60°	1.500	1.000	1.500	23°18'	1.720	0.872	1.500	38°10'	2.143	0.698	1.500	61°17'
85°	6.237	1.000	6.237	92°42'	6.287	0.992	6.237	93°21'	6.485	0.969	6.237	95° 9'

Verzerrungen zu (Pl(n = ...) + MS)/2 (s. Tafel IV)

φ = 0°	1.000	0.818	0.818	11°28'	1.000	0.818	0.818	11° 28'	1.000	0.818	0.818	11°28'
30°	1.000	0.868	0.868	8° 8'	1.162	0.747	0.868	25°12'	0.411	0.615	0.868	46°18'
60°	1.137	1.000	1.137	7°20'	1.468	0.774	1.137	36° 3'	1.950	0.583	1.137	65°20'
85°	4.214	0.986	4.152	76°46'	4.230	0.982	4.152	77° 7'	4.452	0.933	4.152	81°37'

Verzerrungen zu (Pl(n = ...) + W₁)/2 (s. Tafel V)

φ = 0°	1.000	0.818	0.818	11°28'	1.000	0.818	0.816	11°28'	1.000	0.818	0.818	11°28'
30°	1.000	0.912	0.912	5°17'	1.059	0.861	0.912	11°48'	1.159	0.790	0.912	21°36'
60°	1.382	1.000	1.382	18°26'	1.506	0.918	1.382	28° 5'	1.759	0.786	1.382	44°56'
85°	5.538	1.000	5.538	87°14'	5.733	0.967	5.538	90°39'	6.162	0.980	5.538	93° 2'

Verzerrungen zu (Pl(n = ...) + A)/2 (s. Tafel VI)

φ = 0°	0.998	0.816	0.818	11°28'	1.000	0.818	0.818	11°28'	1.000	0.818	0.818	11°28'
30°	1.000	0.891	0.891	6°37'	1.099	0.802	0.877	17°45'	1.387	0.577	0.801	48°44'
60°	1.241	1.000	1.241	12°21'	1.528	0.767	1.171	38°46'	1.846	0.480	0.887	71°53'
85°	4.397	1.000	4.397	78° 1'	9.055	0.980	8.871	107°10'	9.034	0.916	3.696	78° 4'

Gez. v. O. Winkel, Leipzig 1921

Sämtliche Berechnungen von Prof. Dr. Arthur Krause, Leipzig. (Diejenigen zu Eckerts V scheinen bisher nicht veröffentlicht worden zu sein)

Fig. I: Development of Winkels secant cylindrical projection of mean convergence “without” rectangular grid

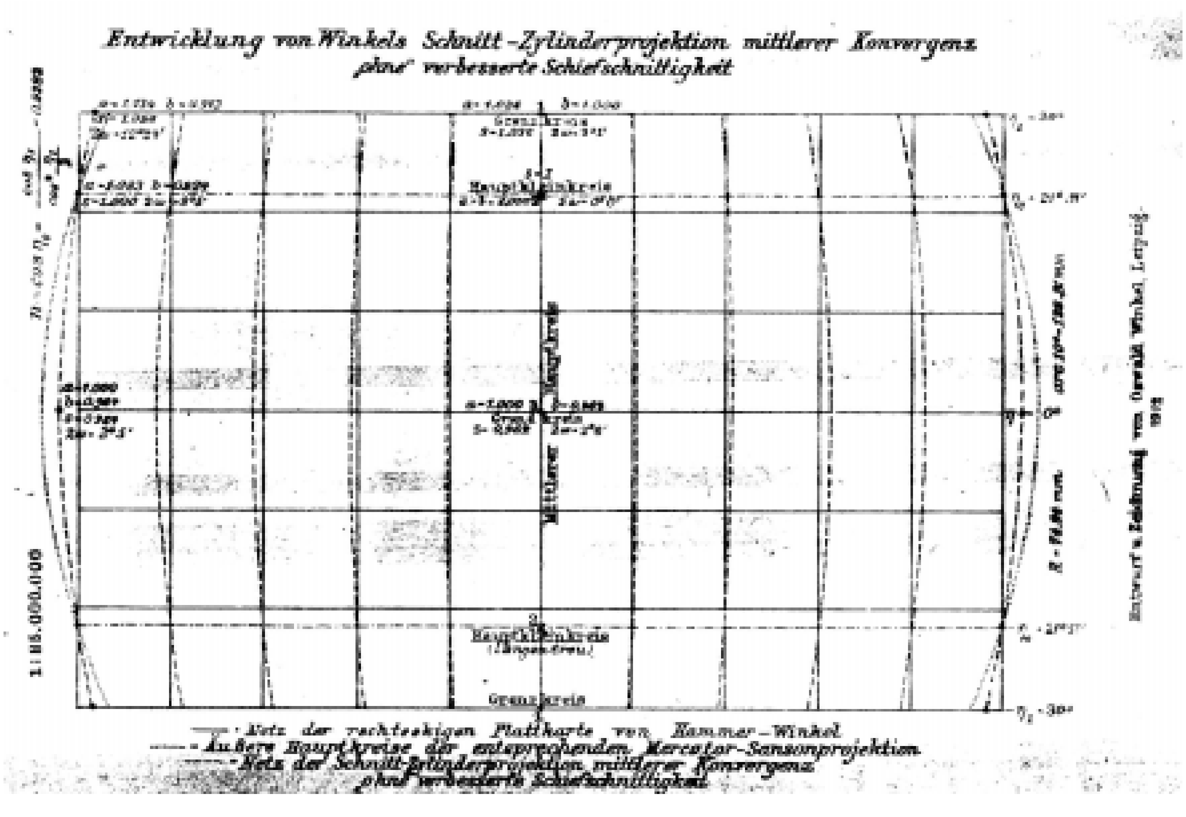

Fig. II: Africa and Europe

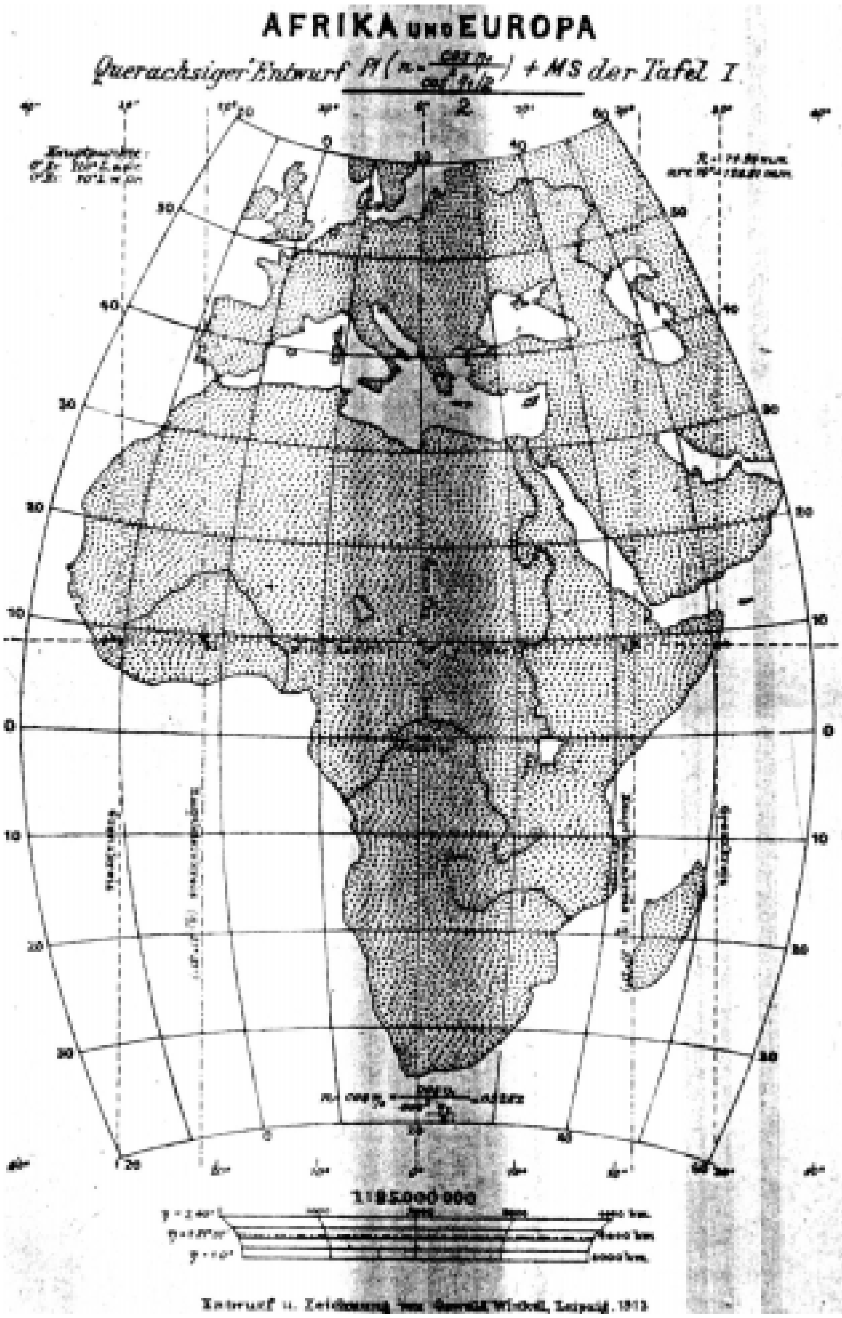

Fig. III: Kilometer graticule for Africa and Europe

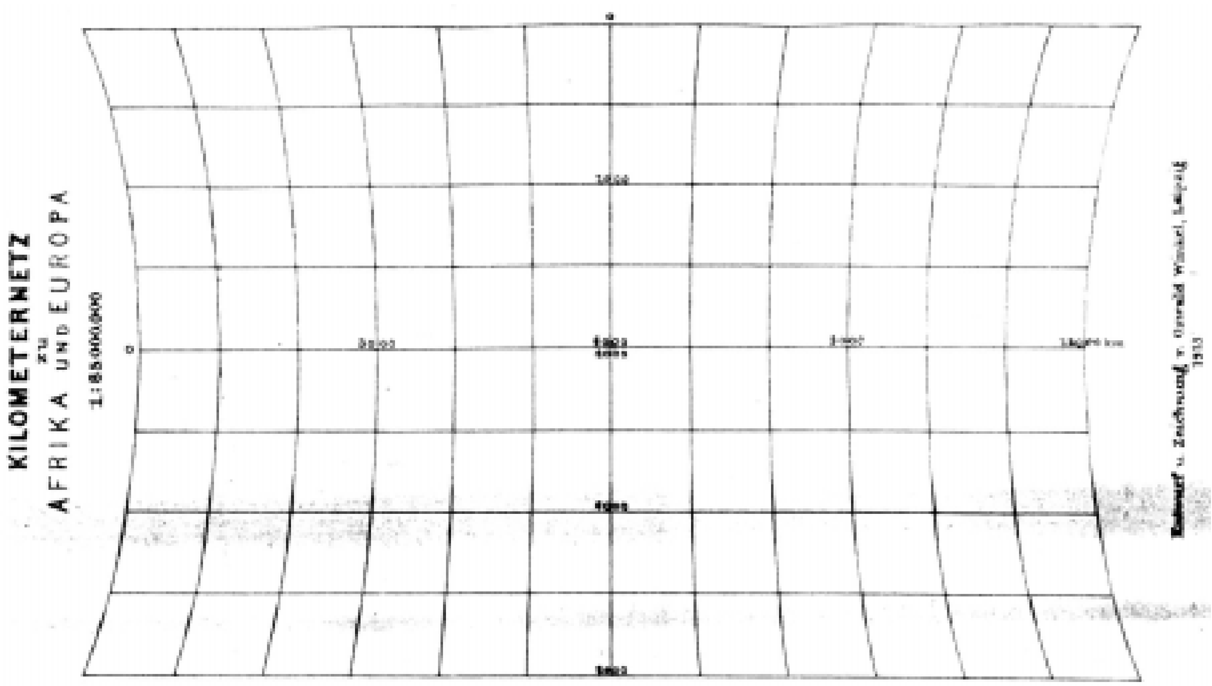

Fig. IV: Winkels secant cylindrical projection of mean convergence (arithmetic average of x, y of the true secant cylindrical projection and Mercator-Sanson projection)

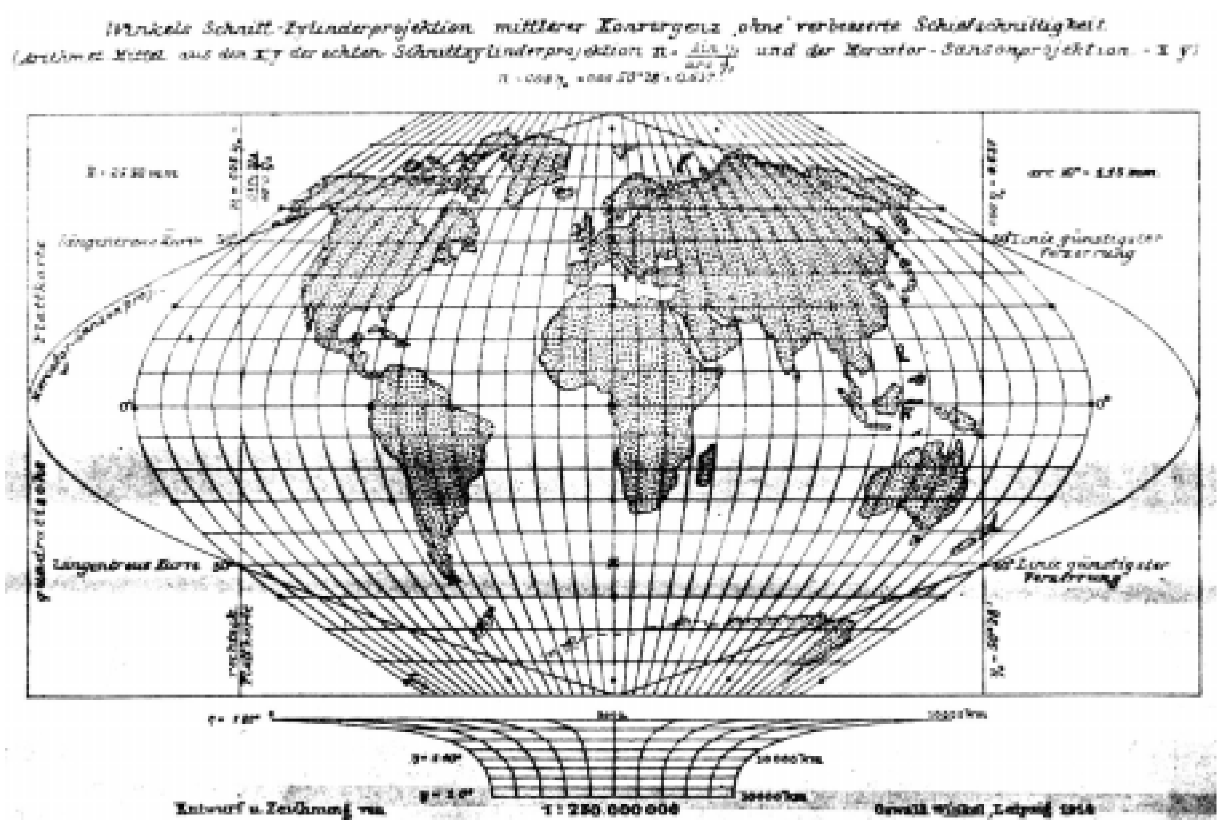

Fig. V: Winkel‘s secant cylindrical projection of mean convergence (arithmetic average of x, y of the true secant cylindrical projection and the equidistant Mollweide planisphere)

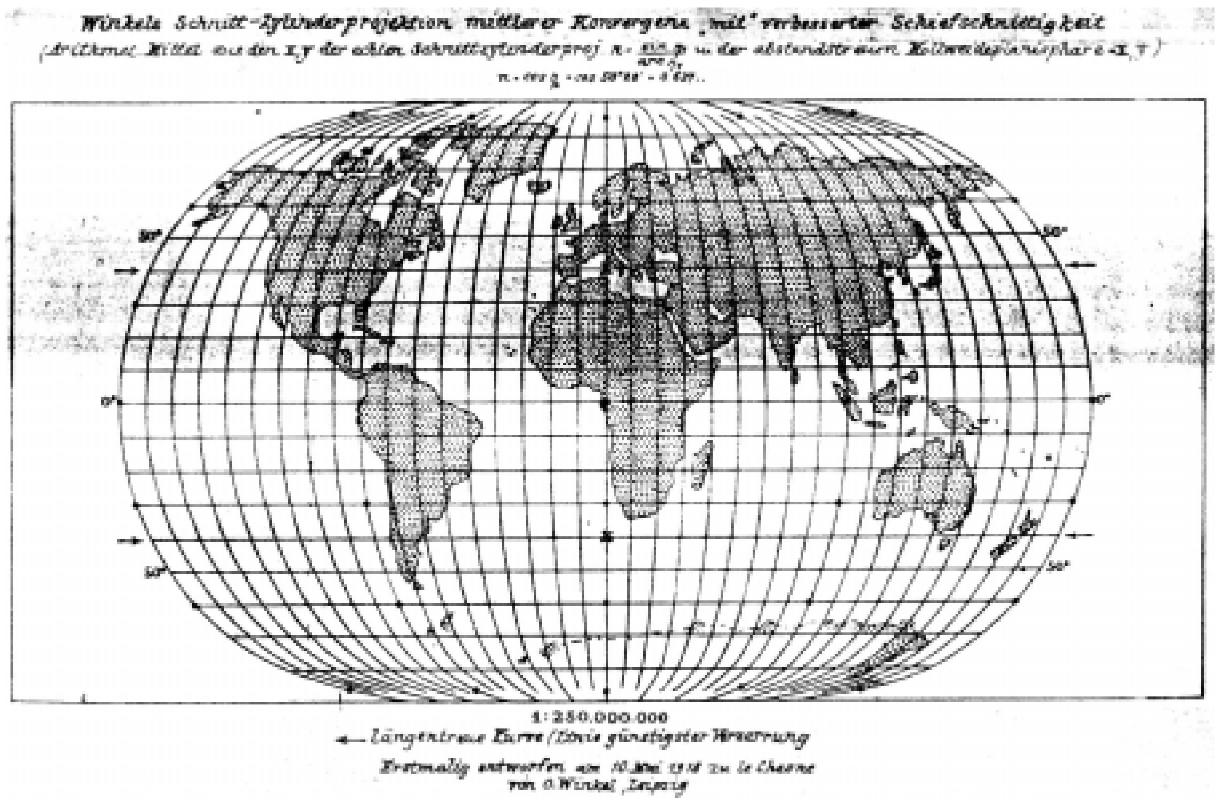

Fig. VI: Winkel‘s cylindrical-azimuthal projection of mean convergence (arithmetic average of x, y of the true secant cylindrical projection and the Aitoff planisphere)