Network analysis is a second major approach to the study of transportation geography. In Chapter 7, the stress was on the understanding of basic forces, such as population and distance, that underlie linkages and flows between nodes. Network analysis, on the other hand, places its greatest stress on the structures formed by these linkages and nodes. The description and analysis of these structures have long been a concern of geographers. Among the measures employed have been the preparation of maps and tables listing distances, capacities, flows, and such indexes as network densities and isochrones, as well as a number of statistically based measures. Network analysis provides another large family of measures that have proven to be extremely useful in the continuing analysis of linkage-and-node systems. The source for these measures is primarily from the mathematical subfields of topology and graph theory. In this chapter, we will first present some generalized full-network measures and then a series of matrices based on measures of the accessibility of individual nodes in the network.

THE NETWORK AS A GRAPH

A major difficulty in describing and analyzing the properties of a network is the overall complexity of the task. In reality, networks are highly complex spatial systems. We must start, therefore, with a substantial simplification of these systems. This permits us to express any system, however complex, as a relatively simple graph. This graph, in turn, can be manipulated so as to express the node-linkage relation in a remarkably large number of ways. This permits us to compare networks with each other and to see just how much, and in what
ways, a particular network has been evolving through time. Initially, therefore, we will confine ourselves to the most basic topological properties of a network and simply ask whether or not a linkage exists between any two nodes. Later, we will consider some of the many other possible properties of those linkages, such as distance or capacity. Once the basic procedures are understood, we could also consider analyzing many other linkage characteristics such as freight rates, scheduling, different types of flow, construction costs, and so forth.

Graph theory, a branch of topology, deals with abstract configurations of points and lines, or nodes and linkages. It does have considerable potential real-world usefulness, however, since it can provide empirical measures of the structural properties of any system once that system is translated as a set of nodes connected by a set of linkages. In graph theory terminology, nodes or points are usually referred to as vertices, and linkages or line segments are usually referred to as edges. We will use nodes and vertices, and linkages and edges, interchangeably in this chapter.

At the highest level of abstraction, transport networks can be represented by a series of vertices (nodes) and a set of edges (linkages). Thus, initially we know only about the presence or absence of linkages between the nodes in a given network. We will start with the full-network measures, the simplest kind, since they provide just one number to measure the connectivity of an entire network. The full-network measures will be followed by a series of matrix-based individual measures in which the network accessibility is represented by a vector of numbers. These accessibility measures are the degree of a node, the $T$- or total accessibility matrix, the Shimbel distance or $D$-matrix, and the $L$-matrix or valued graph. Each will be illustrated by several applications, including the Ohio Interstate highway network, which was used to illustrate the application of spatial interaction models.

**FULL-NETWORK MEASURES—CONNECTIVITY**

When a network is abstracted as a set of edges (linkages) that are related to a set of vertices (nodes), a fundamental question is the degree to which all pairs of vertices are interconnected. This is the connectivity of a network, the degree of connection between all vertices. It is probably the most important single structural property of the network.

**Minimally and Completely Connected Networks**

We will start with the simple network shown in Figure 9.1. This is a minimally connected network. Each vertex (node) is connected to the network, and none is isolated from it. Since it is minimally connected, there are no superfluous edges (linkages). The removal of a single edge would divide the network into two disconnected parts. We can now identify the precise number of edges needed in this network to provide minimal connection. This network of six vertices may be connected by five edges so that none is isolated. This may also be expressed in equation form: $e_{\text{min}} = (V - 1)$; the minimum number of edges needed to create a network is simply one less than the number of vertices in the network.

![Figure 9.1](image)

**Figure 9.1** A minimally connected network. Each vertex (node) is connected to the network. The minimum number of edges (linkages) needed to create a network is $e_{\text{min}} = (V - 1)$, one less than the number of vertices in the network.
the network since a new node is created every time two linkages intersect.
If we were to consider Figure 9.2 as representing a nonplanar graph, such as an airline network in which the flights at any given time would be at different altitudes (or geometric planes), the figure would indeed represent maximal connectivity. If, however, we were to consider it as a planar graph such as a highway network, all linkages would have to be in the same plane and all intersections not corresponding to the original six vertices would have to be eliminated. Figure 9.3 shows how topological maximal connectivity may be calculated for a planar graph. As is shown by the sequence of diagrams, the addition of a new node to any maximally connected network of more than two nodes requires the addition of three new

Figure 9.2 represents the same network, but now it is completely or maximally connected. Every one of the six vertices is directly linked to each of the other five vertices. Now, however, we run into two complications as we try to develop an equation linking this abstract network to real-world transport networks. Our first thought would simply be to say that the number of edges needed for maximum or complete connection is \( V(V - 1) \) or 30 edges (6 \( \times \) 5). This would indeed be true if we were taking direction into account and if the linkage between \( V_1 \) and \( V_6 \) were different from that between \( V_6 \) and \( V_1 \). Such a network is referred to as a digraph or directed graph. If we are not dealing with a digraph and are just interested in the existence or nonexistence of a linkage between \( V_1 \) and \( V_6 \), we can simply divide the \( V(V - 1) \) figure by two. This gives a total of 15 linkages (6 \( \times \) 5)/2, which corresponds with the total edges shown in Figure 9.2. The second complication is a bit less simple to correct. If we were to think of Figure 9.2 as a highway network, there would really be more than six nodes or vertices in

<table>
<thead>
<tr>
<th>Vertices</th>
<th>Maximum number of edges</th>
<th>Diagrammatic representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>△</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>□</td>
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<tr>
<td>5</td>
<td>9</td>
<td>X</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
<td>O</td>
</tr>
<tr>
<td>7</td>
<td>15</td>
<td>O</td>
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</tbody>
</table>

Figure 9.3 A maximally connected planar network. The relationship between the number of nodes (V) and the maximum number of linkages in a planar graph or network is always \( e_{\text{max}} = 3(V - 2) \). The inclusion of each additional node to a network of more than two nodes increases the number of possible linkages by a value of three. There is no intersection of linkages except at a node.
links for the network to again become maximally connected, or $e_{\text{max}} = 3(V - 2)$. Notice that the equation is not accurate if we attempt to apply it to a case where $V = 2$. Maximal connectivity in this case is one edge, but the $e_{\text{max}}$ equation, $3(V - 2)$, would give zero edges. Once $V$ is greater than two, however, the equation gives the correct result. With three nodes, three linkages provide a maximally connected network. In Figure 9.3, three new linkages are needed to connect the fourth new vertex. The same is true of each additional node, and the $e_{\text{max}}$ equation of $3(V - 2)$ gives the same number of linkages as are shown in the drawing. Notice that in the last three diagrams there is now a difference in maximal connectivity between planar and nonplanar graphs. In all instances, the new linkages can only reach three of the preexisting nodes without intersecting existing linkages and creating a new node.

We can now return to the six-node example network and calculate its planar maximal connectivity. As shown in Figure 9.4, this is equal to $3(V - 2)$ or 12, as compared with 15 for the nonplanar case (Figure 9.2). This difference of three is associated with there being, by definition, no nonnodal intersections in the planar or highway example (Figure 9.4) as opposed to several such intersections in the nonplanar or air traffic example (Figure 9.2).

The six-node network can also be used to illustrate another full-network measure of network connectivity, the diameter of a network. The diameter is simply the number of linkages or steps needed to connect the two most remote nodes in the network. The better connected the network, the lower the diameter. In the minimally connected network (Figure 9.1), the diameter is four because it takes four steps to connect $V_1$ to $V_6$, the two nodes most remote from each other. In Figure 9.2, on the other hand, the diameter is only one, because in this nonplanar network, every node is directly connected to every other node and $V_1$ is only one step away from $V_6$, as well as from all other nodes in the network. The maximally connected planar network (Figure 9.4) has a diameter of two since three nodal pairs are two steps away from each other ($V_1$ and $V_3$, $V_2$ and $V_6$, $V_4$ and $V_5$).

We can now return to the stage model of Chapter 1 (Figure 1.33), where we discussed a completely connected network but stopped short of defining just what such a network would look like. Figure 9.5 shows several stages in the development of this same idealized network. The network contains 13 nodes ($V = 13$), so we can identify the number of linkages needed for minimal and maximal connectivity. Since we can assume that we are dealing with a planar network (highway or rail), the number of linkages needed for maximal connectivity is given by $e_{\text{max}} = 3(V - 2)$ or 33. Now as we examine the stages in Figure 9.5, we can see that the first stage of scattered ports (or multiple gateways) and of penetration lines (Figure 9.5a) precedes the development of a minimally connected network. In Figure 9.5, part (b) qualifies as a minimally connected network and parts (c) and (d) fall somewhere between a minimally and maximally connected network.

We can also make a few comments about substantive aspects of the stage model. First, a minimally connected network is unlikely
The Gamma Index

The gamma index, $\gamma$, provides a useful basic ratio for evaluating the relative connectivity of an entire network. It is simply the ratio between the number of edges (linkages) actually in a given network and the maximum number possible in that network: $\gamma = \frac{\text{actual edges}}{\text{maximum edges}}$. The maximum number of edges is called $e_{\text{max}}$. For a planar network, $e_{\text{max}} = 3(V - 2)$. As this simple measure is applied to Figure 9.5, we can establish a scale along which measures of the degree of complete connectivity can be obtained. These range from $(V - 1)/3(V - 2)$, or 0.36, for the minimally connected stage to an index between 0.36 and 1.0 for the last two stages.

We can now move a few steps further and use the gamma index to categorize those networks that fall between minimal and maximal connection. A classification used by engineers consists of three basic network configurations: spinal, grid, and delta.

The spinal configuration is really a minimally connected network. Every node is connected to at least one other node in the network, and it is possible for flow to occur between any two nodes in the network. This flow can only occur along a single path, however, and will usually consist of indirect or multiple linkages. The number of edges necessary for a minimally connected network is $(V - 1)$, one less than the number of vertices in the network. Since the gamma index is $e/e_{\text{max}}$ and the maximum number of edges in a planar network is $3(V - 2)$, the gamma index for a minimally connected network will always be $(V - 1)/3(V - 2)$.

We can also calculate the limit for the spinal gamma index, the value it will approach as the number of vertices become infinitely large. First we may express $(V - 1)/3(V - 2)$ as $\frac{1}{3}(V - 1)/(V - 2)$ or $\frac{1}{3}[V/(V - 2) - 1/(V - 2)]$. As $V$ becomes infinitely large, $V/(V - 2)$ approaches infinity over infinity, or 1, and $1/(V - 2)$ approaches one over infinity, or zero. The equation then becomes $\frac{1}{3}(1) = \frac{1}{3}$ for the upper limit of a spinal gamma index. At the lower end of the range (if $V = 4$), the value for the gamma index will be $\frac{1}{3}$. Thus, the range of values for the spinal network will be between $\frac{1}{3}$ and $\frac{1}{3}$.

The delta network, on the other hand, is characterized by a high density of linkages relative to the number of nodes. Although it
is not a maximally connected network, it consists of several paths or sequences of linkages between pairs of nodes. As shown in Figure 9.6, the delta configuration consists of a triangle linking each set of three nodes. Once the first basic triangle of three nodes is established, two new linkages are required each time a new node is added. Thus, the relationship between nodes and linkages is $2V - 3$, as demonstrated in Figure 9.6. This means that the gamma index will always be $(2V - 3)/3(V - 2)$. Again, we can calculate the limit this time for a delta-configuration gamma index. First we can express

$$\frac{2V - 3}{3(V - 2)} = \frac{1}{3} \left( \frac{2V - 3}{V - 2} \right)$$

$$= \frac{1}{3} \left( \frac{2V}{V - 2} - \frac{3}{V - 2} \right)$$

(9.1)

For networks with an infinitely large number of vertices, $2V/(V - 2)$ approaches 2 and $3/(V - 2)$ approaches zero. Thus, the gamma index approaches $\frac{1}{3}(2 - 0)$ or $\frac{2}{3}$. For smaller networks ($V = 3$), the index will equal 1.0. The range for a delta network is therefore $\frac{2}{3}$ to 1.0.

These limits can now be used to establish precise gamma-index cutoff values for classifying networks such as those shown in Figure 9.5. We can use $\frac{1}{3}$ to $\frac{2}{3}$ as the range for a spinal, or minimal, configuration and $\frac{2}{3}$ to 1.0 as the range for a delta, or maximal, configuration. The third type of network configuration, the grid, is simply transitional between the minimally connected spinal and the maximally connected delta types. Thus, the range of values for the three classical network configurations are

- Spinal: $\frac{1}{3} \leq \gamma < \frac{1}{2}$ where $V > 4$
- Grid: $\frac{1}{2} \leq \gamma < \frac{2}{3}$ where $V > 4$
- Delta: $\frac{2}{3} \leq \gamma < 1.0$ where $V > 3$

In Figure 9.5, therefore, part (b) with a gamma index of 0.36 is a spinal network, part (c) ($\gamma = 0.55$) is a grid network, and part (d) ($\gamma = 0.79$) is a delta network. We could also use these measures to identify times at which different networks attain particular levels, such as spinal and delta configurations, as well as to compare levels of development of networks in different countries or regions.

Figure 9.7 provides another application of the planar gamma index. Ohio’s Interstate highway system with a gamma index of 0.53 would be classified as a grid network. The U.S. highway system, on the other hand, is an older, better connected system and, with a gamma index of 0.70, would be classified as a delta network.

The gamma index can also be applied to nonplanar systems, such as air traffic networks. In the nonplanar case, we find that $\gamma = e/e_{\text{max}} = e/[V(V - 1)/2]$. An example of a nonplanar network was shown in Figure 1.10, the system of morning nonstop flights in the Core region noted in Chapter 1. In this present case, however, the maximum number of flights is $V(V - 1)$ since we will be
accounting all nonstop flights from each of the 26 given origins to however many of the 25 destinations that are connected, so that both i to j and j to i will be included. Thus, the maximum connections would be $26 \times 25 = 650$. In 1966, there were 153 connections, giving a gamma index of $\frac{153}{650} = 0.24$ (Figure 9.9a). In 1993, the connections had increased to 320 (Figure 9.9b), giving a gamma index of $\frac{320}{650} = 0.49$. Thus, in the 27 years of air traffic expansion, 26 large core cities had gone from about one-fourth to about one-half of the maximum possible connections to each other by morning nonstop jet flights.¹

There are many other full-network measures, both planar and nonplanar, all of which provide single-number measures for different network characteristics. Such relatively simple measures, however, can be viewed as only a first step in network analysis. We must now look more carefully at the internal structure of transportation networks.

**INDIVIDUAL MEASURES—MATRIX RELATIONSHIPS**

When examining a transport network for evidence of the spatial organization of an area, a geographer is by no means limited to considering only the full-network characteristics of a network. The emphasis may be placed on the identification of the internal spatial structure of the network's component node-linkage associations. We may examine linkages and flows between centers or we may look at the nodes themselves in terms of their functions and their accessibility to the

¹The nonstop data were collected for the authors by Eric Neubauer.
permit us to consider **five** important aspects of network analysis that are *not* effectively treated by full-network measures:

1. **Placement**: Consideration should be given, not just to the total number of linkages, but also to *where* they are located within a given network.

2. **Direct and indirect linkages**: Direct and indirect linkages should *both* be considered.

3. **Attenuation**: The *differences* between direct and indirect linkages should be treated.

4. **Redundancy**: Corrections should be made for meaningless *round trips*.

5. **Unequal linkages**: In some cases, linkages should be assigned different *weights* instead of assuming that all are equi-valued.

From the full-network measures, we already know that Figure 9.8 is a minimally connected, or spinal, network. Its gamma index indicates that its percent of maximal connection is 42 and that its diameter is 4, since the two most remote nodes are four steps apart.

### Direct Connections—The Degree of a Node

We can now look more carefully at the internal structure of the network and begin to evaluate the accessibility of individual nodes. In the case of Figure 9.8, each linkage is represented by a cell entry of one. A zero in a cell means that there is no linkage between that particular origin node, *i*, and destination node, *j*. Each row therefore consists of a vector for an origin node *i*. The row sums ($\sum_{j=1}^{n} c_{ij}$) for that vector represents the total number of other nodes in the network that are connected to *i* by direct or one-step connections. This provides our first measure of accessibility for each node. For example, $V_3$ is clearly the most accessible node in the network. $V_3$ is directly connected to three of the other nodes, giving it a row sum of three as compared with smaller row sums for each of the other five nodes. This sum of direct linkages is called the *degree of a node* and is the first of the individual measures.
| From     | AK | AL | BA | BO | BU | CH | CI | CL | CO | DA | DE | HA | IN | LO | MI | NY | NO | PH | PI | RI | RO | ST | SY | TO | WA | YO | Total |
|----------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| Akron    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 1   |
| Albany   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 0   |
| Baltimore|    | 1  | 1  | 1  | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 6   |
| Boston   | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    | 13  |
| Buffalo  |    | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 8   |
| Chicago  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |    |    |    |    |    |    | 19  |
| Cincinnati|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 4   |
| Cleveland|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 8   |
| Columbus |    | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 4   |
| Dayton   |    | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 3   |
| Detroit  | 1  | 1  | 1  | 1  | 1  |    | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 9   |
| Hartford |    | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 3   |
| Indianapolis|    | 1  |    |    |    |    |    |    |    |    |    |    |    | 1  |    |    |    |    |    |    |    |    |    |    |    | 4   |
| Louisville|    | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 3   |
| Milwaukee|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 1   |
| New York | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |    |    |    |    |    |    | 22  |
| Norfolk  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 1   |
| Philadelphia| 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    | 11  |
| Pittsburgh|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 6   |
| Richmond |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 1   |
| Rochester| 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 3   |
| St. Louis|    | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 8   |
| Syracuse |    | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 4   |
| Toledo   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 2   |
| Washington| 1  | 1  | 1  | 1  | 1  |    | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 7   |
| Youngstown| 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 2   |

Total 3 0 4 6 5 19 7 9 6 4 7 4 5 4 4 4 22 0 7 11 0 8 3 8 3 2

(a)

**FIGURE 9.9** Morning nonstop matrices in 1966 and 1993. A comparison of the 1966 (a) and 1993 (b) matrices of morning jet nonstops shows the strengthened role of Chicago and the tendency for the networks to expand from a concentration on just two cities to a more even distribution of morning nonstops.

We now return to the morning jet nonstops in the U.S. Core and compare the 1966 and 1993 networks expressed as matrices (Figure 9.9). As noted in Chapter 1, the row sums permit us to rank the cities far more accurately than was possible simply by visual inspection of a map. From the gamma index we already know that the total accessibility has doubled. Now we can use the changes in the degree of each node to evaluate air network change in the Core during the 27-year period. (Refer again to footnote 1.)

Chicago and New York still rank as the two best-connected cities in the network. Again, however, there is evidence of the persistence of Chicago's historic gateway role. Chicago moved ahead of New York to rank as the best connected of the Core cities with morning nonstops to all the other cities ex-
cept Toledo and Norfolk, which were probably both affected by traffic shadow (Detroit and Washington, DC, respectively).

The most striking change between 1966 and 1993, however, has been the weakening of the earlier concentrations on the larger two or three centers. In 1966, New York and Chicago dominated the network. They were the only cities connected to more than 13 of the other 25. Nearly half the cities were connected to fewer than four other cities. By 1993, only three cities had fewer than four connections, and 14 of them were connected to more than half of the other cities. Certain hub cities, such as Pittsburgh and Cincinnati, showed particularly dramatic jumps in ratings.

We can also consider nonstops on a world scale. Figure 9.10 shows nonstop connections between a set of major world cities in the mid 1960s. Again, it is difficult visually to compare how well each city is connected to the others in the network. When the network is expressed as a matrix, however, and the sums for each row vector are tabulated, the cities may be ranked according to the degree-of-a-node measure. By this measure, London was the world’s most accessible city.
since it had nonstop jet connections to more major world cities than did any other city. Paris was a close second, and Rome, Copenhagen, and New York were in third place. By the mid-1970s, New York had increased its nonstop connections enough to tie Paris for second place. Rome, Athens, Miami, and Moscow shared third place. Since then, London, Paris, and New York have ranked fairly consistently as the world’s most accessible cities.

The degree of a node has serious limitations, however, as a measure of accessibility and should be regarded as merely an early step in the process of developing a set of more versatile measures. For surface modes of transport, accessibility involves more than the direct connections between nodal pairs. We are usually interested in knowing the accessibility between nodes that takes into account the second of the five network characteristics, indirect connections, or linkages between nodal pairs that pass through one or more intermediate centers. For example, if we were to attempt to rank Core cities in terms of accessibility within the Interstate Highway System, the node with highest degree would be Indianapolis, which has seven direct Interstate highway links to other major cities. However, all the major cities are connected to each other by indirect linkages, and these must also be taken into account in any evaluation of comparative accessibility. Indianapolis, for example, is connected to Cleveland by a sequence of three links: Indianapolis–Dayton, Dayton–Columbus, and Columbus–Cleveland. Any evaluation of Indianapolis’s total accessibility to the entire
network must take account of similar indirect linkages to all the other core cities. Even in the case of airline traffic, where total direct or nonstop linkages are meaningful indicators of relative accessibility, the degree of a node is a poor discriminator because of the importance of one-stop or connecting flights that should also be taken into account.

The Total Accessibility Matrix \( (T) \)

The degree of a node tells us more than the gamma index because it does take into account the placement of the linkages in the network. Now, however, we can proceed to the second of the five important network characteristics we wish to consider: the inclusion of both direct and indirect linkages.

Again, the matrix format is useful. Now the process of matrix multiplication can be used to produce a series of matrices that do show the number of indirect connections or paths between individual nodes as shown in Figure 9.11.

Matrix multiplication involves the element-by-element multiplication of the rows in one matrix by the columns of another. For example, to derive a value for the cell of the first row and first column of a matrix \( (c_{11}) \), we multiply the first row times the first column. That is, we multiply the first element in row one by the first element in column one. Then we multiply the second element in row one by the second element in column one and so on. Finally, we take the sum of all the element-by-element multiplications and record that sum as the value of the cell in the first row and first column \( (c_{11}) \) of a new matrix.

In evaluating network connectivity, we will be multiplying the connectivity matrix, which we will now call \( C^1 \), by itself (Figure 9.11). Each entry in the \( C^1 \) matrix represents the presence (1) or absence (0) of a direct, or one-step, linkage.

We can now illustrate how the matrix multiplication procedure has meaning in terms of the indirect, or two-step, connection between \( V_1 \) and \( V_3 \). To get this connection we must power the \( C^1 \) matrix or multi-

![Figure 9.11](image_url)
ply it by itself to produce a $C^2$ matrix, $C^I \times C^I = C^2$. The $V_1$ to $V_3$ connection is represented by cell $c_{13}^2$ in the $C^2$ matrix. Figure 9.11 shows the element-by-element process. Five of the six element-by-element multiplications result in zero because one of the linkages has no direct connections. In case of the $c_{12}^1 \times c_{23}^1$ multiplication ($1 \times 1 = 1$), the positive result indicates that a two-step connection does exist between $V_1$ and $V_3$. The $C^1$ matrix shows a one in $c_{12}^1$, indicating that you can go directly from $V_1$ to $V_2$ in one step, and a one for $c_{23}^1$, an indication that you can then go directly from $V_2$ to $V_3$ in one step. Each positive element-by-element multiplication, therefore, represents one two-step path connecting $V_i$ and $V_j$. In the case of the $c_{13}^2$ cell in $C^2$ indicating two-step linkages between $V_1$ and $V_3$, the sum of these values is one, indicating that there is way of going between $V_1$ and $V_3$ in exactly two steps. The cell value in $c_{13}^2$ may therefore be expressed in this case as $c_{12}^1 \times c_{23}^1$ or more generally as $c_{ij}^2 = \sum_k c_{ik}^1 c_{kj}$, with $k$ representing the intermediate nodes over which the linkage is made.

The entire matrix $C^2$ in Figure 9.11 now provides a complete enumeration of the two-linkage paths in the network. For $V_1$, for example, there is one two-step linkage to itself (to $V_2$ and back), no two-step linkages to $V_2$, one to $V_3$, and none to $V_4$, $V_5$, or $V_6$. Note that the $C^2$ matrix shows only those linkages with exactly two steps. All direct or one-step linkages such as those between $V_1$ and $V_2$ are represented by zeros in the $C^2$ matrix. Also note that the matrix powering procedure does not distinguish between round-trips ($c_{ij}^2$) and origin–destination ($c_{ij}^2$) trips. All entries on the main diagonal of the matrix represent round-trips. For example, the number three in cell $c_{33}^2$ indicates that there are three ways of going from node $V_3$ back to itself (via $V_2$, $V_4$, or $V_5$).

We still have not considered all the indirect linkages, however. The linkage between $V_1$ and $V_5$ is a three-step linkage, one that passes through two intermediate centers, just as in the case of the Indianapolis–Cleve-

land Interstate highway linkage that passes through Dayton and Columbus. The same logic applies, however, to even longer or more attenuated indirect linkages, and we can again turn to matrix multiplication and now produce a three-step connection matrix, $C^3$.

The $C^3$ matrix is obtained simply by multiplying the $C^2$ matrix by the $C^1$ matrix (Figure 9.12). The entry for a three-step, $V_1$ to $V_5$, connection now registers a one, indicating that there is one way to move between $V_1$ and $V_5$ in exactly three steps. In $C^1$, the nonzero entry for cell $c_{12}^1$ shows that there is a one-step linkage from $V_1$ to $V_2$; and, in $C^2$, the nonzero entry for cell $c_{25}^2$ shows that there is a two-step linkage from $V_2$ to $V_5$. Multiplied together they give a three-step connection, a nonzero entry in the $C^3$ matrix. Thus, the entire $C^3$ matrix provides a complete enumeration of the number of ways in which the nodes in the system are connected to each other in exactly three steps. Notice that round-trips are just as prevalent in the $C^3$ matrix as they were in $C^2$ even though all the main diagonals are now zero, since there is no way in this minimally connected network that you can move from $i$ to $i$ in exactly three steps. Now the round-trips are to be found in the off-diagonal elements as parts of some redundant ways of moving from $i$ to $j$. For instance, there are two ways you can go from $V_1$ to $V_2$ in three steps: $V_1$ to $V_2$, $V_2$ back to $V_1$, and $V_1$ to $V_2$ again; $V_1$ to $V_2$, $V_2$ to $V_3$, and $V_3$ back to $V_2$. Both of these involve redundancies, or meaningless round-trips. Nonetheless, they provide measures of the overall degree of connectedness of the network and are clearly related to the general accessibility of individual nodes.

After computing the $C^3$ network, we must again decide whether or not to keep on powering the matrix. We still have not accounted for linkages between all nodes in the network. $V_1$ and $V_6$, the two most remote points in the network, are four steps apart. Notice that in all three matrixes the $V_1$ to $V_6$ cell registers a zero ($c_{16}^1$, $c_{16}^2$, $c_{16}^3 = 0$). We must therefore power the matrix one
more time ($C^1 \times C^3 = C^4$) to have nonzero values in all cells. This does provide us with a useful decision rule as to when we can stop powering the connection matrix. We should power it until the two most remote points are linked to each other. More concisely, we should power to the diameter of the network.

We now have a measure of nodal accessibility but it is a rather cumbersome one, since it consists of four separate matrices (Figure 9.13). We can add these matrices up, however, and produce a useful single summary matrix that does take account of both direct and indirect linkages, the second of the five important network characteristics discussed earlier. This summary matrix is called the total accessibility matrix, or $T$-matrix, and is also shown in Figure 9.13.

The matrix addition involved in developing the $T$-matrix is a more straightforward procedure than matrix multiplication. For example, the 3 in the $V_1$ to $V_2$ cell of the $T$-matrix represents the sum of $V_1$ to $V_2$ entries in the four connection matrices: one in $C^1$, zero in $C^2$, two in $C^3$, and zero in $C^4$. Thus, the entry in cell $l_{12}$ indicates that there are a total of three ways of moving between $V_1$ and $V_2$ in four steps or fewer. The three ways are the single-step connection ($c^1_{12}$) and the two three-step connections that involve roundtrips ($c^3_{12}$). The $T$-matrix row sum of 14 indicates that there are 14 ways in which you can move from $V_1$ to all nodes in the network, including movements from $V_1$ to itself. The larger this row sum is, the more accessible that node. The most accessible node in Figure 9.13 is therefore $V_3$. There are 38 ways of going from $V_3$ to all nodes in the network in four steps or fewer. This is more than for any other node in the network, indicating that $V_3$ is better connected to the rest of the network than is any other node. In the case of this small network, it is, of course, quite evident that $V_3$ is the most accessible node, but as networks become more complex, it quickly becomes impossible to make such judgments simply by visual inspection. With the help of comput-
ers, however, the matrix-powering procedures are just as feasible for large systems with many nodes as for this example, and it becomes quite possible to rank nodal accessibility with relative precision.

The $T$-matrix can also be used to evaluate the accessibility of the entire network by adding up all the row sums. There are a total of 142 ways in which the six nodes shown are connected to all the nodes in the network in four steps or fewer. We know, however, that this measure is made somewhat less meaningful by the number of redundancies or meaningless round-trips. For example, the row sum of 38 for $V_3$ includes 14 ways of going from $V_3$ to $V_3$ in four steps or fewer. One way to reduce the number of such trips in the matrix is simply to eliminate the main-diagonal cells ($t_{ii}$) that represent trips for $V_i$ to $V_i$.

For example, we can reduce the row sum of 38 for $V_3$ by 14 so that it becomes 24. We can now say that there are 24 ways in which we can move from $V_3$ to all other nodes in four steps or fewer. Since there are a total of 40 main diagonal entries in the $T$-matrix, we can subtract this from 142, the sum of the row sums, and arrive at a new total of 102 as a measure of the total connectivity of the entire network. This says that in the network, there are 102 ways in which we can move from each node to each other node in four steps or fewer.

We can now return to the Ohio Interstate highway system, discussed in Chapter 7, to show how the $T$-matrix could be used by a planner or decision maker to help solve a specific highway question: What would be the best linkage to add to this existing inter-

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**FIGURE 9.13** The total accessibility or $T$-matrix. The $T$-matrix is the sum of the connection matrix $C^3$ and all the matrices that enumerate direct or multistep paths between nodes in the network,

$$T = C^1 + C^2 + C^3 + \ldots + C^n,$$

where $n$ is the diameter of the network.
state system? Figure 9.14 shows three proposed new linkages: Cleveland–Dayton, Columbus–Marietta, and Toledo–Columbus.

A first step in deciding between the three new linkages could be to use the $T$-matrix in evaluating the entire network and the individual nodal accessibilities of the cities in the system.

In this case, the gamma index provides the decision maker with no help. In all three cases, one link is added, increasing the gamma index from 0.56 to 0.61. The diameter is three in the original network and in each of the three proposed additions. The degree of node is not useful since it simply provides the sum of direct linkages for each node, and it does not consider the many indirect linkages in the system. The $T$-matrix provides a better means of evaluating the effects of the three different placements of a proposed new linkage on both direct and indirect linkages.

The $T$-matrix total (Figure 9.15) for the original network indicates that there are 254 ways of going from all nodes in the network to all nodes in the network in three steps or fewer (222 ways of going from all nodes to all other nodes). Columbus ranks as the most accessible city, with 50 ways of going to all cities in the network. Cleveland is second with 46, followed by Cambridge with 37, due to its strategic location at the intersection of Interstates 70 and 77, and Dayton with 37 also. The outlying positions of Marietta and Youngstown were reflected by their low rankings.

According to the $T$-matrix, the Cleveland–Dayton addition would have the greatest effect (Table 9.1). It would result in a 37 percent increase of $T$-matrix totals in the number of ways of going from each node to each node in the system as compared with a 33 percent increase for the Toledo–Columbus addition and a 25 percent increase for the Columbus–Marietta addition. A Cleveland–Dayton addition would move Cleveland ahead of Columbus as the most accessi-

![Diagram of Ohio Interstate highway system with proposed additions](image)

**FIGURE 9.14** The Ohio Interstate highway system (a) and three proposed additions: Cleveland–Dayton (b), Columbus–Marietta (c), and Toledo–Columbus (d).
FIGURE 9.15 Ohio Interstate highway system T-matrix and city ranking. The T-matrix total shows that there are 254 ways of going from all nodes in the system to all nodes. Columbus ranks as the most accessible city in the original network with 50 ways of going to all cities in three steps or less.

<table>
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<th>TOL</th>
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<th>YNG</th>
<th>CAM</th>
<th>MAR</th>
<th>COL</th>
<th>CIN</th>
<th>DAY</th>
<th>Total</th>
<th>Rank</th>
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<td>4</td>
<td>1</td>
<td>9</td>
<td>6</td>
<td>5</td>
<td>37</td>
<td>3</td>
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254

TABLE 9.1 The Ohio Interstate System and Proposed Additions—T-Matrix Rankings

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<tr>
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<th>Original Network</th>
<th>CLE–DAY Addition</th>
<th>COL–MAR Addition</th>
<th>TOL–COL Addition</th>
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<td>DAY</td>
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<td>3</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
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<tr>
<td>MAR</td>
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<td>8</td>
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<td>8</td>
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<tr>
<td>Total</td>
<td>254</td>
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<td>318</td>
<td>338</td>
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The T-matrix can also be used to consider the effects on nodal accessibility of different networks, such as those associated with different modes of transportation. As was noted earlier, the Interstate Highway System represents a move away from the concepts that governed federal highway policies before 1956. Previous policy emphasized the construction of a relatively dense network of highways linking urban centers of all sizes with each other and linking urban centers with their hinterlands. The Interstate Highway System is more comparable to the airline and rail network in that it links large metropolitan centers with each other via a set of trunklines. The new highway network, however, will not necessarily focus on the same set of cities as either the railroad or airline networks. Figure 9.16 compares the Interstate Highway System and the railroad system in the southeastern United States. In a classic study representing the first major use of graph theory to study a transportation network, William Garrison used a modified T-matrix to evaluate the comparative accessibility of cities in the two networks. To reduce the effects of attenuation whereby each linkage was at first weighted the same, whether it represented a first step or a fifth step, Garrison introduced a scalar weighting procedure into the matrix multiplication process. The scalar had to be less than 1.0 so as to reduce the weighting of each successive step as the number of linkages increases:

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**FIGURE 9.16** Interstate Highway and rail accessibility in the southeastern United States. Scalar-weighted $T$-matrix values were calculated for southeastern cities in both the Interstate Highway and the rail systems. Atlanta is the most accessible center in both networks, but Birmingham, Montgomery, and Savannah are less accessible on the highway network than on the rail network.

$$T = \sum_{i=1}^{n} s^i C^i$$

$$= s^1 C^1 + s^2 C^2 + s^3 C^3 + \ldots + s^n C^n$$

The scalar value used was $s = 0.3$. Thus, a direct connection is weighted by $s^1$ or 0.3; a two-linkage connection is $s^2$ or 0.09; a three-linkage connection is weighted by $s^3 = 0.027$, and so forth. Thus, in the scalar-weighted $T$-matrix, those connections requiring many linkages would not be as heavily weighted as they would be in the unweighted $T$-matrix. As shown in Table 9.2, Atlanta clearly ranks as the most accessible node on both the highway and the rail network. Cities such as Montgomery, Savannah, and Birmingham, however, which occupy strategic positions on the rail network, are considerably less accessible in the highway network. On the other hand, cities such as Asheville, Spartanburg, and Statesville have markedly improved accessibility rankings on the Interstate Highway System as compared with the railroad network.

**The Shimbel Distance or $D$-Matrix**

To some extent, the use of a scalar in the matrix multiplication process does reduce the effect of attenuation, the third of the five important network characteristics listed. The impact of a given linkage is inversely related to the number of steps that preceded it. Attenuation is not completely removed, however, and there really are no accepted ground rules for the choice of scalar value. The 0.3 used in the southeastern U.S. study had to be arbitrary. Nor does the scalar-weighted $T$-matrix treat the critical fourth
TABLE 9.2 A Comparative Ranking of Cities in the Southeast According to Scalar-Weighted Accessibility to the Interstate Highway System and the Railroad Network

<table>
<thead>
<tr>
<th>RANK</th>
<th>INTERSTATE HIGHWAY SYSTEM</th>
<th>RAILROAD NETWORK</th>
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<tr>
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<td>Atlanta</td>
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<td>Columbia</td>
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<td>Spartanburg</td>
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The problem of redundancy. Even if we remove the main diagonal so as to eliminate all \( c_{ii} \) (the ways of moving from \( i \) back to \( i \)), we are left with many off-diagonal examples of trips that pass through a given node more than once.

One way to eliminate both attenuation and redundancy from the analysis of a network is to use a matrix developed by Shimbel for the study of communication networks.\(^3\)

In the Shimbel distance or \( D \)-matrix, we are interested not in the total number of paths

between any two nodes but in the one shortest path between them, the one involving the fewest linkages. Rather than record how many ways there are for us to move between $i$ and $j$, we now simply record the number of linkages involved in the shortest path between them.

The elements of the $D$-matrix indicate the "distance" (in number of linkages) of the shortest path between all pairs of nodes in a network. Operationally, it is simply a bookkeeping procedure carried on while powering the $C$-matrix to its diameter. Each time the $C$-matrix is powered, we keep track of any new nonzero elements. If there is a new nonzero element, the power of the matrix is entered into the appropriate row and column of the $D$-matrix. The first two steps in the Shimbel procedure for the six-node example are illustrated in Figure 9.17. The $C^1$ connection still shows all direct connections as one, all others as zero. The $D^1$ matrix also shows the direct connections as one. In this case, however, the number one in cell $d_{ij}$ is to be interpreted as indicating that the shortest path between that $i$ and $j$ is a direct or single-step linkage. The main diagonal ($d_{ii}$) elements are all zero as they are in $C^1$. In the case of the $D$-matrix, however, the main diagonal elements stay at zero through all subsequent powerings since the shortest path between $i$ and $i$ is still zero linkages. For the same reason, all values of one in the $D^1$ matrix also remain unchanged through subsequent powering, since the shortest paths for each of those nodal pairs has already been established as one. The other off-diagonal elements represent indirect connections, those involving more than one step. At this stage they are simply left blank (indicated by a dash in Figure 9.18). They will be filled in later, when the $C$-matrix powering procedure indicates that they are connected. The shortest paths for $V_3$, for example, are one-step connections to $V_2$, $V_4$, and $V_5$, but we still do not know the shortest paths to $V_1$ and $V_6$.

We now move to the $D^2$ matrix and record only the new connections that are registered in the $C^2$ matrix. For each new connection in the $C^2$ matrix, the number two is recorded in the appropriate cell of the $D$-
matrix. All other values in the $C^2$ matrix are ignored. For example, the $V_3$ row in the $C^2$ matrix (Figure 9.17) now shows that there is one way of going from $V_3$ to $V_1$ and from $V_3$ to $V_6$ in two steps. The number two is therefore entered in both the $d_{31}^2$ and the $d_{36}^2$ cells.

This bookkeeping process continues until the network is powered to its diameter of four, as shown in Figure 9.18. The $D^4$ matrix is the Shimbel distance or $D$-matrix. No off-diagonal cell remains to be filled at this point, and each cell entry indicates the shortest path (fewest linkages) between the $i$ and $j$ that define that cell. The row sums now tell us the minimum number of steps required to connect node $i$ to all other nodes in the network. For example, the row vector for $V_3$ in the $D$- (or $D^4$) matrix shows that $V_3$
can reach $V_1$ in two steps; $V_2$, $V_4$, and $V_5$ in one step each; and $V_6$ in two steps. This gives a row total of seven, indicating that all other points in the network can be reached from $V_3$ in seven steps. On the other hand, it takes 13 steps to reach all other points from $V_1$ since that node is three steps away from $V_4$ and $V_5$ and four steps away from $V_6$. Notice that the $V_1$ to $V_6$ cell was not filled until the $C^4$ network was calculated since the four steps between $V_1$ and $V_6$ define the diameter of the network, the number of linkages needed to connect the two most remote nodes, $i$ and $j$.

Thus, $V_3$, the node with the lowest row sum ($\sum_{j=1}^{n} d_{ij}$) in the $D$-matrix, is the most accessible node in the network; $V_1$ and $V_6$, the nodes with the highest row sums, are the least accessible nodes in the network. As in the case of the $T$-matrix, the sum of these row sums ($\sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij}$) provides a single-number measure of the accessibility of the entire network. The total of 62 in Figure 9.18 indicates that each node in the network can reach each other node in a minimum total of 62 steps. Improvements in the network would reduce this total. The $D$-matrix is free from the redundancies of the $T$-matrix and does account for attenuation by reducing the importance of multilinkage connections.

We can now return to the Ohio Interstate highway system and apply the $D$-matrix to the problem of deciding between the three proposed additions (Table 9.3). The total number of steps required for each city to reach every other city is 104 in the original network. It only requires 10 steps for either Columbus or Cleveland to reach every other city; 11 steps for Cambridge. As in the case of the $T$-matrix, Youngstown and Marietta rank as the least accessible cities. In general, the distinctions between cities are considerably less marked in the $D$-matrix. The many redundancies associated with the $T$-matrix results in an exaggeration of the distinction between nodes. The difference between the least accessible and the most accessible nodes in the $D$-matrix is far less than in the $T$-matrix.

$D$-matrix totals indicate that the Columbus–Marietta addition would provide a greater improvement in accessibility than either of the other two. The total number of steps would be reduced from 104 to 98, as opposed to 100 for the Cleveland–Dayton addition and 102 for Toledo–Columbus. Thus, there is a discrepancy between $D$-matrix results and the $T$-matrix results, which indicated a greater accessibility increase for the Cleveland–Dayton addition. The greater clustering of cities around Cleveland leads to more round-trips and more ways of moving between cities, thereby increasing $T$-matrix measures. If we were to simply think of reducing the minimum number of steps or between-city trips needed to connect all nodes with each other, the $D$-matrix total indicates that the Columbus–Marietta connection would establish Columbus as the most accessible city just ahead of Cleveland, and Marietta would move ahead of Youngstown, which would then rank as the least accessible city.

We can also compare $D$-matrix results with scalar-weighted $T$-matrix analysis of the southeastern U.S. highway system. As shown in Table 9.4, Atlanta is the most accessible center in both measures, and such centers as Macon, Columbia, Augusta, and Florence also continue to rank high in accessibility. The spatial pattern based on the $D$-matrix minimum-distance path, however, shows a stronger spatial organization focusing on Atlanta. Centers having direct connection to
this center have high accessibility rankings. Conversely, centers such as Statesville, Winston-Salem, Charlotte, and Asheville are lower ranking on the basis of minimum paths. These lower rankings reflect the elimination of the redundancies in the scalar-weighted T-matrix. The D-matrix minimum path procedures were less sensitive to the spatial clustering in North Carolina.

The D-matrix can also be used to measure the changes of a network through time. Table 9.5 shows how the Amtrak system evolved from its initial network in 1971 through the early years of its existence. The network viewed in terms of 18 major cities showed a clear improvement in accessibility during its first years as selective additions and deletions were made in the network. In 1971, it took a total of 708 steps for all cities in the network to reach all other cities. Three years later the total had dropped to 648, a decline of nearly 10 percent. Chicago retained its historic rail centrality, ranking as the most accessible city in the network through the period. In both 1971 and 1974, it took only 24 steps to connect Chicago with the other 17 Amtrak cities in the network. Three other historically important railroad centers—St. Louis, New Orleans, and Washington, DC—improved their accessibility more rapidly than the entire network and as a result were much closer to Chicago in 1974. A fourth historic rail gateway, Cincinnati, also improved its accessibility more rapidly than the entire network to move up from eighth to sixth place. The greatest changes, however, were associated with some additional routes allocated to Norfolk and Tampa. Both cities had accessibility improvements of nearly 30 percent, moving up from the bottom of the list to near the middle.

The Valued Graph or L-Matrix

Thus far, we have developed network measures that have addressed four of the five shortcomings noted in the simple full-network measures, such as seen in the gamma index. Placement as well as direct and indirect linkages are accounted for by the total accessibility or T-matrix and attenuation and redundancy by the Shimbels distance or D-matrix. Now we can consider a matrix that also addresses the fifth measurement problem: unequal linkages.

All matrices up to this point have been based on topological distance. Each linkage was recorded as a one, regardless of its actual length. For many purposes this is quite satisfactory as, for example, when we are in-
<table>
<thead>
<tr>
<th>Network 1</th>
<th>Network 2</th>
<th>Network 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>3. New Orleans</td>
<td>St. Louis</td>
<td>Washington, DC</td>
</tr>
<tr>
<td>5. St. Louis</td>
<td>New Orleans</td>
<td>San Francisco</td>
</tr>
<tr>
<td>Kansas City</td>
<td>Kansas City</td>
<td>New Orleans</td>
</tr>
<tr>
<td>Miami</td>
<td>San Francisco</td>
<td>Cincinnati</td>
</tr>
<tr>
<td>Seattle</td>
<td>10. Houston</td>
<td>10. Seattle</td>
</tr>
<tr>
<td>Detroit</td>
<td>Miami</td>
<td>Miami</td>
</tr>
<tr>
<td>Boston</td>
<td>14. Tampa</td>
<td>12. Tampa</td>
</tr>
<tr>
<td>458</td>
<td>49 (+11)</td>
<td>49</td>
</tr>
<tr>
<td>17. Norfolk</td>
<td>16. Buffalo</td>
<td>Norfolk</td>
</tr>
<tr>
<td>18. San Diego</td>
<td>49 (+11)</td>
<td>49</td>
</tr>
<tr>
<td>708</td>
<td>55</td>
<td>55</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Network 4</th>
<th>Network 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>July 1971</td>
<td>January 1974</td>
</tr>
<tr>
<td>1. Chicago</td>
<td>1. Chicago</td>
</tr>
<tr>
<td>2. Washington, DC</td>
<td>2. St. Louis</td>
</tr>
<tr>
<td>3. Cincinnati</td>
<td>New Orleans</td>
</tr>
<tr>
<td>4. New York</td>
<td>Washington, DC</td>
</tr>
<tr>
<td>New Orleans</td>
<td>New York</td>
</tr>
<tr>
<td>Kansas City</td>
<td>Cincinnati</td>
</tr>
<tr>
<td>St. Louis</td>
<td>Miami</td>
</tr>
<tr>
<td>8. San Francisco</td>
<td>Tampa</td>
</tr>
<tr>
<td>9. Miami</td>
<td>San Francisco</td>
</tr>
<tr>
<td>Seattle</td>
<td>Kansas City</td>
</tr>
<tr>
<td>11. Buffalo</td>
<td>11. Houston</td>
</tr>
<tr>
<td>Tampa</td>
<td>12. Seattle</td>
</tr>
<tr>
<td>15. Los Angeles</td>
<td>Detroit</td>
</tr>
<tr>
<td>Norfolk</td>
<td>16. Buffalo</td>
</tr>
<tr>
<td>17. Boston</td>
<td>Boston</td>
</tr>
<tr>
<td>664 (-19)</td>
<td>664 (-19)</td>
</tr>
<tr>
<td>708</td>
<td>683 (-25)</td>
</tr>
</tbody>
</table>

Source: David Hill, "A Graph Theory Analysis of AMTRAK," unpublished term paper, Department of Geography, Ohio State University, 1974.

Interested in the structure or the degree of connectedness of the nodes and linkages in a network. In many cases, however, we are interested in the actual distances between nodes. For most types of passenger and commodity traffic we are concerned with actual distance, which in turn may be expressed in miles, in cost, or in time-cost.

The valued graph, or L-matrix, the third matrix we will consider, permits us to incorporate unequal linkages in any way we choose—distance, cost, time-cost—or even
such things as traffic flow or construction costs.

As shown in Figure 9.19, we again start by expressing the network as a matrix, relying on the familiar six-node network as our example. This time, however, we will insert the average driving time in the network between each adjacent pair of nodes. The overall structure of the resulting $L$-matrix is identical to the initial connection $C$-matrix, but the individual cell entries are different. First, we weight the direct linkages between the nodes by the driving time needed to travel over them. Second, we insert zeros along the main diagonal, because there is zero time involved in the connections of a node to itself. Third, we enter a value of infinity in each cell where there is no direct or one-step linkage.

As was true in the case of the $C$ and the $D$ matrices, the $L$ matrix presents the same information that can be read directly from the six-node diagram.

We now can move from the initial valued graph $L^1$ to the $L^2$ graph showing two-step connections by visual inspection of the diagram in Figure 9.19. It obviously takes 30 minutes to go from $V_1$ to $V_3$ in two steps and 35 minutes to go from $V_3$ to $V_5$ in two steps, and it is not possible (time = infinity) to go from $V_1$ to $V_6$ in two steps. What is not so obvious, however, is the type of matrix-powering procedure that would permit us to quickly identify multistep paths in complex, real-world networks where visual determination would be impossible.

With three relatively minor modifications, the matrix-powering procedure used to generate the $T$-matrix can be adapted to the generation of an $L$-matrix consisting of shortest paths between each $i$ and $j$ in the network as shown in Figure 9.20. First, we must substitute addition for multiplication in the element-by-element, row-times-column procedure. We add the driving times between points just as we would in reading a road map and estimating mileage between two points by adding the mileage on the intervening segments. Second, we look for the minimum value of the element-by-element operations rather than adding them up. We are interested in the shortest distance between each $i$ and $j$ and not the total distances on all possible paths. Finally, we follow a cumulative procedure in recording shortest paths in the final $L$-matrix just as we did in the $D$-matrix.

We can now apply these modifications to the indirect connection between $V_1$ and $V_3$ (Figure 9.20). As evident from the infinity symbol in the $l_{13}$ cell ($V_1$ to $V_3$) of the initial valued-graph matrix, $L^1$, no direct linkage exists between $V_1$ and $V_3$. For the two-step linkage, we carry on the element-by-element, row-by-column addition shown in the diagram. We are simply considering the additive relationships between $V_1$ and all other nodes in the network, and then between each of those nodes and $V_3$. Taking row 1 and column 3 of the $L^1$ matrix, we have

![Network Diagram](image)

**FIGURE 9.19** A network as a valued graph or $L^1$ matrix. The abstraction of a network as a graph and its representation as a matrix is not limited to the topological properties of the network. This example makes use of average driving time over individual linkages. This information may be recorded in the appropriate cell of the $L^1$ matrix. If no direct connection exists between a nodal pair such as $V_1$ and $V_3$, the value of infinity ($\infty$) is recorded in the cell.
Two-step connection \( v_1 \rightarrow v_3 \)

\[
(v_1 \rightarrow v_1) + (v_1 \rightarrow v_3) = 0 + \infty = \infty \\
(v_1 \rightarrow v_2) + (v_2 \rightarrow v_3) = 10 + 20 = 30 \\
(v_1 \rightarrow v_3) + (v_3 \rightarrow v_3) = \infty + 0 = \infty \\
(v_1 \rightarrow v_4) + (v_4 \rightarrow v_3) = \infty + 10 = \infty \\
(v_1 \rightarrow v_5) + (v_5 \rightarrow v_3) = \infty + 30 = \infty \\
(v_1 \rightarrow v_6) + (v_6 \rightarrow v_3) = \infty + \infty = \infty
\]

Minimum cost path is 30

FIGURE 9.20 Indirect connections in a valued graph or \( L \)-matrix. To determine the value of a two-step linkage between \( v_1 \) and \( v_3 \), for example, each element of the third column of the valued graph matrix is added to each element of the first row of the same matrix. From this element-by-element addition, the minimum sum is determined and inserted in the appropriate cell of the new matrix, \( L^2 \).

\((0 + \infty), (10 + 20), (\infty + 0), (\infty + 10), (\infty + 30), \) and \((\infty + \infty)\). Now instead of adding these as was done in the \( C \)-matrix, we simply select the minimum value, which is 30, and record it in the \( l_{13} \) cell as the shortest path between \( v_1 \) and \( v_3 \). At this point it is important to note that this shortest path passes through \( v_2 \). Even though it is not recorded in the \( L \)-matrix, we will make use of this value later in reconstructing the actual minimum path. Since we are following a cumulative procedure, the minimum travel time of 30 between \( v_1 \) and \( v_3 \) stays in the \( l_{13} \) cell right up to the final \( L \)-matrix.\(^4\)

The procedures leading to the final \( L \)-matrix, or valued graph, are shown in Figure 9.21. As in the case of the previous matrices, we have computed the matrix to the diameter. The \( L^2 \) matrix was added to by the \( L^3 \) or three-step matrix, which has entries in all cells except \( l_{16} \), representing the \( v_1 \) to \( v_6 \) connection. Since this is a four-step connection defining the diameter of the network, it is necessary to calculate an \( L^4 \) matrix. This

\(^4\)In more complex networks, it is possible that there could be a shorter travel time for a connection involving more steps, in which case \( l_{13} \) would be replaced.
matrix shows a travel time of 65 minutes for the $V_1$ to $V_6$ (and $V_6$ to $V_1$) cells and is the final $L$-matrix, or valued graph, since there are no more infinity values recorded.

The elements in the $L$-matrix represent the minimum distance measured in travel time between every pair of nodes. The row sums ($\Sigma_{j=1}^{n} l_{ij}$) in the matrix represent the total travel time needed to go from point $i$ to every other point $j$ in the network (starting from $i$ each time). Again $V_3$ is clearly the most accessible node in the network. It only takes a total of 125 minutes to travel from $V_3$ to all the other points in the network as compared with 205 minutes from $V_1$ and $V_6$, the least accessible points in the network.

A comparison of the results of the $L$-matrix (Figure 9.21), or valued-graph procedure, with those based on Shimbel’s $D$-matrix (Figure 9.18) indicates strong structural similarities. Since the $L$-matrix is based on minimum travel time and the $D$-matrix on minimum number of linkages, there will, of course, be major differences in the values that compose the final vector of nodal accessibility. The resultant rankings, however, may or may not be different. For example, $V_3$ is the most accessible node in each case and $V_1$ and $V_6$ the least accessible. On the other hand, the effect of the weightings can be seen in nodes $V_4$ and $V_5$. In the $D$-matrix of minimum linkages, $V_3$ is the more acces-
sible. It takes nine steps for \( V_5 \) to reach all other points, as opposed to 11 for \( V_4 \). In the L-matrix of minimum travel times, the weighting of the linkages makes \( V_4 \) the more accessible, 165 minutes to reach every other point, as opposed to 185 minutes for \( V_5 \).

Again, the sum of the row sums \((\Sigma l_{ij})\) provides a single number measure of accessibility of the entire network as was true of both the T- and the D-matrices. In this case, the total is 1050 minutes. Thus, it would take a total of 1050 minutes (17 hours 30 minutes) to travel from every point in the network to every other point.

Note how similar the final matrix is to the familiar matrix of highway distances that accompanies most road maps. In both cases, the minimum distance (or travel time) may be read off from the matrix cell representing the intersection between the origin and the desired destination.

We can now summarize more formally the procedures we used for developing a valued graph or L-matrix of minimum travel times in a simple, minimally connected, six-node network. Those same procedures, when generalized, can be used to develop similar matrices of minimal travel times between the more numerous nodes of far more complex networks, where visual inspection is of little use.

We used a set of Boolean rules to establish the needed modifications of matrix power in a more general form. Each cell \( l_{ij} \) in an L-matrix is no longer the sum of the products of the links from each origin \( i \) to all intermediate points \( k \) and from each intermediate point \( k \) to all destination points \( j \), as represented by the equation introduced with the C-matrix: \( \Sigma_{k=1}^{n} l_{ik} \times l_{kj} \). Instead, each cell is the minimum value of the sums of these two-stage \((l_{ik} \text{ and } l_{kj})\) linkages from origin \( i \) to \( k \) and then from \( k \) to destination \( j \), as represented by the equation \( \min(l_{ik} + l_{kj}).\)

In the example used in Figure 9.20, the two-step linkage between \( V_1 \) and \( V_3 \), all possible travel times between \( V_1 \) and \( V_3 \) were added, and the linkage with the minimum travel time was the two-step connection from \( V_1 \) to \( V_2 \) and from \( V_2 \) to \( V_3 \). Thus, \( \min(l_{ik} + l_{kj}) \) is \( l_{12} + l_{23} \), or 10 plus 20. \( V_1 \) is \( i \), \( V_3 \) is \( j \), and \( V_2 \) is the \( k \)-value or intermediate point, which represents the shortest path. In the course of the computations, \( k \)-values for each linkage are stored for later use.

We can now apply the L-matrix to the problem of the additional link to be added to the Ohio Interstate highway system. The initial T-matrix analysis suggested that the Cleveland-Dayton addition would bring about the greatest increase in accessibility. That is, the Cleveland-Dayton addition would permit us to go from each point in the network to each other point in the network in more different ways than either of the other proposed additions. The D-matrix measures, however, suggested that the Columbus-Marietta addition would result in the greatest improvement of accessibility. The Columbus-Marietta addition would permit us to go from each point to every other point in fewer total steps than either the Cleveland-Dayton or Toledo-Columbus additions. This seemed to be a better guide in this case since the D-matrix is free from the redundancies and attenuations embedded in the T-matrix.

Now, however, we can look more critically at the individual steps, or between-city linkages, which were used to construct the D-matrix. As shown in Figure 9.22a, these linkages are clearly unequal in distance (and therefore in driving time). An L-matrix (9.22b) thus permits us to evaluate the comparative accessibility improvements of the three proposed additions in terms of mini-

\(^5\)In the D-matrix, the sequence of linkages \( V_1 \) to \( V_2 \) to \( V_3 \) to \( V_4 \) is equal to the sequence \( V_1 \) to \( V_2 \) to \( V_3 \) to \( V_5 \). Two intermediate nodes were involved in both cases, and the total number of linkages was three. This structural relationship does not change with the L-matrix. The travel-time for the last segments of the two linkage sequences does differ, however. The \( V_3 \) to \( V_4 \) connection is no longer equal to the \( V_3 \) to \( V_5 \) connection. It is 10 minutes compared with 30 minutes, and \( V_4 \) has a travel-time of 20 minutes less than \( V_5 \) from \( V_2 \).

\(^6\)In the Boolean transformation, multiplication is replaced by addition, that is, \( l_{ik} \times l_{kj} \) is replaced by \( l_{ik} + l_{kj} \), and addition is replaced by minimization, or \( \Sigma_{k=1}^{n} (l_{ik} \times l_{kj}) \) is replaced by \( \min(l_{ik} + l_{kj}) \).
FIGURE 9.22 The Ohio Interstate highway system and the \( L \)-matrix. Since the distances between cities on the map (a) are unequal, the valued graph, or \( L \)-matrix, is an appropriate form of analysis. The matrix (b) shows the minimum distance between cities. Comparison of the total minimum distances associated with the proposed additions (c) show that the Toledo-Columbus addition would result in the greatest savings.
mizing the total travel distance needed to go from each city to every other city.

The $L$-matrix totals (Figure 9.22b) tell us that it would take 9792 miles (196 hours at 50 miles per hour) to go from each city in the network to every other city in the original network. Although the city rankings on the $D$-matrix (Table 9.3) and the $L$-matrix (Figure 9.22c) are similar, there are some differences associated with the unequal lengths of the Interstate linkages. In the $D$-matrix rankings, Cleveland was tied with Columbus as the most accessible city. Since the Interstate segments around Columbus and Cambridge are shorter than those around Cleveland, both of those cities moved ahead of Cleveland in the $L$-matrix ratings. Similarly, Marietta moved up in the rankings while Toledo dropped to seventh place.

The $L$-matrix also suggests that the third alternate addition, Toledo–Columbus, would result in a greater increase in accessibility than either the Cleveland–Dayton or Columbus–Marietta additions, suggested by the $T$-matrix and the $D$-matrix, respectively (see Figure 9.22c). The Toledo–Columbus addition would reduce the total distance needed for each city in the system to reach every other city by 2.5 percent from 9792 miles to 9544 miles, as compared with 0.5 percent for the Columbus–Marietta addition and 0.4 percent for the Cleveland–Dayton addition. This savings of approximately five hours of total driving for all cities with the Toledo–Columbus addition, as opposed to the original network, is, of course, based on only a single vehicle traversing the network. Actually, the five hours should be multiplied by the thousands of vehicles traversing the system every day.

As we attempt to evaluate the merits and shortcomings of these three measures, the $L$-matrix does seem to be the most useful matrix for this particular purpose. Despite the many redundancies, however, the $T$-matrix is clearly a more sensitive measure, showing improvements of 20 or 30 percent with the additional linkages as opposed to improvements of less than 10 percent in the other two cases. In cases where redundancies might be regarded as desirable, the $T$-matrix could perhaps be preferred. In this case, the $D$-matrix, although more useful than the $T$-matrix, also does not seem to be as useful as the $L$-matrix. If we had been dealing with domestic air traffic, however, it would have been more useful because the number of connections (or linkages) is usually more important than is their length. That no stops are required at each node, however, indicates that the unequal length of the Ohio Interstate linkages is quite significant and that the $L$-matrix is therefore the best of the three measures for this problem.

At this point, however, the $L$-matrix results themselves should be given a closer look. By no means do they represent the end of the complex decision processes associated with most transport policy problems. We are still well short of being able to make a realistic choice between a Toledo–Columbus, Cleveland–Dayton, and Columbus–Marietta addition. We have already observed that it is necessary to consider traffic flow if we are to properly evaluate the magnitude of the effects of the additions. This suggests a weakness in all three matrix operations since they take no account of flow. We should therefore add a traffic flow component not just to the total driving time but to each linkage. This would call for a scalar weighting of each $l_{ij}$ operation by traffic flow. Comparison of original network row sums with the row sums of the two additions would then more accurately indicate the comparative savings in vehicle-hours. There would still be a problem, however, in that no Interstate flow figures would be available for the new linkages. Here, we might turn to a spatial interaction or gravity model analysis of flow such as was discussed in Chapter 7.

If we were to incorporate the populations of the eight cities into a spatial interaction model such as discussed Chapter 8, with a distance exponent ($\beta$) equal to 2.0, the results (Table 9.6) would reinforce the $L$-matrix result, which showed the Toledo–Columbus addition as providing the greatest increase in total accessibility. The 1.49 percent increase for Toledo–Columbus is con-
TABLE 9.6 Rankings of Proposed Ohio Interstate Additions Based on a Spatial Interaction Analysis

<table>
<thead>
<tr>
<th>Original Network</th>
<th>Cle-Day Addition</th>
<th>Col-Mar Addition</th>
<th>Tol-Col Addition</th>
</tr>
</thead>
<tbody>
<tr>
<td>City</td>
<td>Rank</td>
<td>Expected Flow</td>
<td>Rank</td>
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<tr>
<td>CIN</td>
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<td>911.8</td>
<td>1</td>
</tr>
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</tr>
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</tr>
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<td>7.9</td>
<td>8</td>
</tr>
<tr>
<td>TOTAL</td>
<td></td>
<td>3856.7</td>
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</tr>
</tbody>
</table>

Considerably greater than the 0.31 percent increase for Cleveland-Dayton. The Columbus-Marietta addition ranks third with only a 0.14 percent increase over the original network. City rankings, however, would not be much affected by any of the additions. The spatial interaction model is much affected by the clustering of large, metropolitan population centers in the southern and western parts of the Ohio Interstate network. The rankings are essentially the same for the original network and each of the three proposed additions. Cincinnati, Dayton, and Cleveland are the highest ranking; Marietta and Cambridge are the lowest ranking.

Thus, the result of the inclusion of population in the spatial interaction measure, which is closely related to the expected traffic flow, would enable supporters of the Toledo-Columbus addition to claim the greatest savings in total vehicle-hours. Still further information would be needed, however, before a policy decision could be reached. Columbus-Marietta supporters could argue for using the criterion of vehicle-hours saved per mile of new construction since both the Toledo-Columbus and Cleveland-Dayton additions are considerably longer. This could broaden the discussion to an inclusion of costs, and it would not take supporters of the other two proposals long before they called attention to the more difficult Appalachian terrain associated with the Columbus-Marietta addition.

As such discussion is carried on, however, it is important to note the role of network analysis. Various types of network analysis can provide useful estimates of the benefits of alternate transport route proposals, if the analysis is properly modified to suit local circumstances. These benefits, such as comparative savings in total vehicle-hours, can in turn then be matched with expected costs. The decision maker may then consider the resulting cost-benefit ratio in making the final decision. This decision, however, will still be affected by additional intangibles related to the social, economic, or political impact of the proposed addition, as we have seen in the historical discussions in Chapters 3 and 4. The decision maker, however, will now be aided in making a final decision by being better able to weigh choices suggested by these intangibles against any possible consequences in the form of less favorable cost-benefit ratios. How much will it cost to choose the more expensive alternative, and are the social-political advantages worth the price?

We can now expand our scale of observations and move from the analysis of a single state, first, to a large region (the U.S. Core), then to the entire country.

The Interstate connections to the cities of the U.S. Core region are shown in Figure 9.23. This network can be abstracted as a graph of 56 nodes connected by 202 linkages. For a D-matrix, we can prepare a
Ten most accessible cities

**FIGURE 9.23** $D$-matrix accessibility rankings in the U.S. Core for the Interstate Highway System. The numbers at each city represent $D$-matrix rankings based on the minimum number of steps needed to go from each city to each of the other 55. The 10 most accessible cities by this measure are strongly clustered in the center of the Core, with a single outlier at Syracuse, New York.

A $56 \times 56$ matrix in which each cell entry records the presence or absence of a direct linkage between every pair of nodes in the network. Table 9.7 records the $D$-matrix row sums for each of the 56 cities, and their rankings are shown on the map. Cleveland ranks as the most accessible city in the Core according to the Shimbel $D$-matrix measure. It is possible to go from Cleveland to each of the other 55 cities in the region in 192 steps. Each step in this case represents an Interstate linkage between two of the cities. Madison, Wisconsin, is the least accessible city in the system, requiring fully 378 steps to reach all the other cities. Figure 9.23 shows the 10 most accessible cities according to the $D$-matrix. They are strongly clustered in Ohio in the center of the region, with a single outlier at Syracuse.

Table 9.8 shows the accessibility rankings when an $L$-matrix or valued graph is applied to the driving times between the nodes. Each linkage is weighted by an average driving time of 55 miles per hour. Figure 9.24 shows that Cleveland again ranks as the most accessible city in the Core with a row sum indicating that you can drive from Cleveland to each of the other 55 cities in a total of 261 hours. The location of the entire group of 10 top-ranking cities, however, shows a significant change. The $L$-matrix, which is sensitive to the abundance of short linkages in the eastern part of the Core, reveals a spatial organization of high accessibility extending along an axis, similar to the Main Street Trunkline, from Cleveland in the west to Harrisburg, Pennsylvania, in the east. Cities located on or near this axis have the best highway proximity to all the major urban centers of the Core.

We can now use the $L$-matrix to evaluate node accessibility on the Interstate Highway System for the entire United States. One hundred U.S. cities are ranked according to valued-graph accessibility based on average driving time, as shown in Figure 9.25. In the $L$-matrix, the row for each city $i$ consists of 99 cells, each containing the driving time from $i$.
TABLE 9.7  D-matrix Accessibility Rankings of Core Cities on the Interstate Highway System

<table>
<thead>
<tr>
<th>Rank</th>
<th>City</th>
<th>Shortest-Path Accessibility</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Cleveland</td>
<td>192</td>
</tr>
<tr>
<td>2</td>
<td>Columbus</td>
<td>206</td>
</tr>
<tr>
<td>2</td>
<td>Toledo</td>
<td>206</td>
</tr>
<tr>
<td>2</td>
<td>Youngstown</td>
<td>206</td>
</tr>
<tr>
<td>5</td>
<td>Cincinnati</td>
<td>210</td>
</tr>
<tr>
<td>6</td>
<td>Syracuse</td>
<td>215</td>
</tr>
<tr>
<td>7</td>
<td>Akron</td>
<td>220</td>
</tr>
<tr>
<td>7</td>
<td>Dayton</td>
<td>220</td>
</tr>
<tr>
<td>9</td>
<td>Fremont</td>
<td>221</td>
</tr>
<tr>
<td>10</td>
<td>Cambridge</td>
<td>223</td>
</tr>
<tr>
<td>11</td>
<td>St. John</td>
<td>225</td>
</tr>
<tr>
<td>12</td>
<td>Pittsburgh</td>
<td>228</td>
</tr>
<tr>
<td>12</td>
<td>Indianapolis</td>
<td>229</td>
</tr>
<tr>
<td>12</td>
<td>Scranton</td>
<td>229</td>
</tr>
<tr>
<td>15</td>
<td>Louisville</td>
<td>231</td>
</tr>
<tr>
<td>16</td>
<td>Lexington</td>
<td>233</td>
</tr>
<tr>
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<td>Charleston</td>
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<tr>
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<td>Detroit</td>
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</tr>
<tr>
<td>24</td>
<td>Breezewood</td>
<td>256</td>
</tr>
<tr>
<td>25</td>
<td>Danbury</td>
<td>262</td>
</tr>
<tr>
<td>26</td>
<td>Staunton</td>
<td>265</td>
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<tr>
<td>27</td>
<td>Champaign</td>
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<td>28</td>
<td>Stroudsburg</td>
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<tr>
<td>29</td>
<td>Marshall</td>
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<tr>
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<td>Hagerstown</td>
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</tr>
<tr>
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<td>Norristown</td>
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</tr>
<tr>
<td>43</td>
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<tr>
<td>45</td>
<td>St. Louis</td>
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</tr>
<tr>
<td>46</td>
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</tr>
<tr>
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</tr>
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<tr>
<td>53</td>
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</tr>
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<td>Milwaukee</td>
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<tr>
<td>55</td>
<td>Madison</td>
<td>378</td>
</tr>
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TABLE 9.8  L-matrix Accessibility Rankings of Core Cities on the Interstate Highway System

<table>
<thead>
<tr>
<th>Rank</th>
<th>City</th>
<th>Shortest time-path from the city to all others (hours)</th>
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<tbody>
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<tr>
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</tr>
<tr>
<td>53</td>
<td>Moline</td>
<td>455.6</td>
</tr>
<tr>
<td>54</td>
<td>Madison</td>
<td>471.9</td>
</tr>
<tr>
<td>55</td>
<td>St. Louis</td>
<td>479.0</td>
</tr>
<tr>
<td>56</td>
<td>Boston</td>
<td>496.2</td>
</tr>
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</table>
Ten most accessible cities

**FIGURE 9.24** L-matrix accessibility rankings in the U.S. Core for the Interstate Highway System. The value recorded for each city is the shortest total driving time from that city to all others in the network. The 10 most accessible cities now lie along an east-west axis similar to the Main Street Trunkline.

**FIGURE 9.25** U.S. L-matrix accessibility ranking for the Interstate Highway System. The 10 most accessible cities form a large cluster in the east-central part of the country. Louisville in the single most accessible city.
to the particular city $j$ represented by that column. Louisville is the single most accessible city. You can reach all 99 of the other cities in a total of 1741 hours of driving time from Louisville. The cities of the Pacific Northwest are the least accessible, with total driving times in excess of 4000 hours. The 10 most accessible cities on the Interstate Highway System form a large cluster in the eastern and central part of the country. The cluster is focused on Kentucky and extends into five adjacent states. It is not located in the geometric center of the country because the Interstate Highway System is much denser and better connected in the East.

As we did on a regional basis in the Southeast, we can now compare road and rail networks, this time for the entire country. Figure 9.26 and Table 9.9 are based on the application of the $L$-matrix to rail distances between the same set of nodes over major rail linkages. Indianapolis ranks as the most accessible city. In a total of 91,000 miles of travel over major rail linkages, you can go from Indianapolis to each of the other 99 cities in a total of 91,000 miles. Louisville is not far behind with 92,000 miles. The contour maps of rankings show few differences between the road and rail maps. The historical rail focus on Chicago pulls the rail contour for the top 10 farther north, and, to a lesser extent, Memphis pulls the rail contour farther south. In the West, the rail focus on Los Angeles keeps southern California out of the bottom 10.

For the most part, however, the two maps are more notable for their similarities than for their differences. There are two reasons for this. One is the scale, or the level of spatial aggregation. At the state scale, if we had compared Ohio Interstate highway and rail networks, we would have noted sharp differences. At the regional scale in the Southeast, there were a number of significant differences in ranking; at the national scale, these differences are smoothed over and we see two mature well-developed networks, each with its densest concentration in the East. A second reason for the similarity lies in the country’s spatial organization around the primary trunklines, which were first identified in Chapter 1. As was noted there and in the discussion of the evolution of U.S. transportation in Chapters 3 and 4, there has been a tendency in U.S. transport development at the national scale for new systems of different modes to develop along the same trunklines, thereby reinforcing the existing spatial organization.

Air traffic flow has a similar pattern of geographic concentration. Figure 9.27 shows the area in the middle of the country where 10 of the 15 large airports with more connecting than originating traffic were located. This central zone is remarkably similar to the zones of greatest road and rail accessibility as represented by the contours for the 30 most accessible cities.

Finally, we might compare Figures 9.25 and 9.26 with another measure of overall

<table>
<thead>
<tr>
<th>INTERSTATE HIGHWAYS</th>
<th>RAILROADS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Rank</strong></td>
<td><strong>City</strong></td>
</tr>
<tr>
<td>1</td>
<td>Louisville</td>
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<tr>
<td>2</td>
<td>Nashville</td>
</tr>
<tr>
<td>3</td>
<td>Indianapolis</td>
</tr>
<tr>
<td>4</td>
<td>St. Louis</td>
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<td>Knoxville</td>
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<td>Cincinnati</td>
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<td>Chattanooga</td>
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<td>Chicago</td>
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<td>Toledo</td>
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<td>Memphis</td>
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<td>Bristol</td>
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<td>Birmingham</td>
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<td>Muskegon</td>
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<td>22</td>
<td>Moline</td>
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<tr>
<td>23</td>
<td>Greenville</td>
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<tr>
<td>24</td>
<td>Kansas City</td>
</tr>
<tr>
<td>25</td>
<td>Pittsburgh</td>
</tr>
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</table>
accessibility discussed in Chapter 7. The potential map (Figure 7.15) is an accessibility measure that shows proximity to an aggregate population. The potential map differs from the highway and rail accessibility maps primarily in being weighted by population (or some other type of node magnitude), since it is based on a gravity or spatial interaction model. This shifts the zone of maximum accessibility or proximity northeastward into the Core region where the largest cities are concentrated. In addition, the most remote region, which is the West Coast on the highway and rail maps, shifts into the Intermontane West on the potential map since there are fewer large cities there.

**SUMMARY**

In this chapter, elementary concepts of graph theory have been used to describe and measure some basic properties of networks. When applying graph theory to the analysis of a network, we idealized and simplified the network as a graph. By treating only the topological properties of a network, we were able to describe the structure in a relatively simple fashion. Although a number of full-network measures could have been derived for this purpose, we chose one, the gamma index, for both planar and nonplanar networks. By establishing limits for this index, it was possible to identify three basic network configurations: spinal, grid,
FIGURE 9.27 Fleming’s airport centrality zone compared with highway and rail centrality zones. Fleming delimited an area in the east-central part of the country where most of the large airports with more connecting than originating traffic were located. This zone is similar to the zones of greatest road and rail accessibility, which are shown on the map by dashed and dotted lines, respectively, for the 30 most accessible cities. After: Douglas K. Fleming and Yehuda Hayuth, “Spatial Characteristics of Transportation Hubs: Centrality and Intermediacy,” Journal of Transport Geography, 2 (1994), p. 11.

and delta. These measures could, in turn, be used to compare different networks as well as to trace more precisely the development of a particular network through time, as was exemplified by applying the gamma index measures to the idealized stage model of network development discussed in Chapters 1 and 3.

We then developed a series of matrices that permitted us to go beyond the highly generalized full-network analysis of a given network and to critically examine its internal structure. We identified five important aspects of network analysis that are not effectively treated by single-number full-network measures such as the gamma index. These involve the consideration of (1) placement of the linkages with respect to each other; (2) both direct and indirect linkages; (3) attenuation, the difference between single-step and multistep linkages; (4) redundancies, or meaningless round-trips; and (5) different weights for unequally valued linkages according to their distance cost or some other criteria.

When the network was expressed as a matrix, a connection matrix, $C^n$, was formed, and the row sums $\Sigma_{j=1}^{n} C_{ij}^1$ could be referred to as the degree of a node. This mea-
sure took account of the placement of nodes within the system and provided a measure of accessibility for each node, namely, the number of direct connections it had to the other nodes. A single-number measure of the accessibility of the entire network could then be calculated by summing the total direct connections from all nodes to all other nodes. The matrices for multistep connections could be calculated by powering the C-matrix to \( C^n \), where \( n \) is the diameter of the network or the number of steps between the two most remote nodes of the network. These \( n \) C-matrices were then summed to form a T-matrix, which provided measures of accessibility involving both direct and indirect linkages. T-matrix values for a particular cell, \( ij \), indicated the number of ways there are of going from \( i \) to \( j \) in \( n \) steps or fewer. Row sums indicated the number of ways of going from \( i \) to all nodes in the network (including \( i \) itself), and the totals of the row sums indicated the total number of ways of going from each to each node in the network. If the main diagonal cell values are subtracted, the totals indicate the number of ways of going from every node in the network to every other node in the network. The higher the total, the better connected the network is considered to be.

The Shimbel distance or D-matrix addressed both attenuation and redundancy since it consisted of the shortest paths (in terms of number of steps or linkages) between all nodes in a network. Thus, a cell value in the D-matrix indicated the least number of linkages between a given \( i \) and \( j \). The row sums indicated the number of steps needed to connect a given node to all other nodes, and the total row sums indicated the total number of steps needed to connect every node in the network with every other node. The lower the total, the better connected the network is considered to be.

The last consideration, unequally valued linkages, was addressed by the L-matrix, or valued graph. The L-matrix makes it unnecessary to restrict the concept of a distance metric to one that is purely topological. Values that measure actual distance, cost, or driving time between nodes may be used in an L-matrix analysis. The network was still abstracted as a graph and represented as a matrix. The cell entries of the matrix, however, now represented some measure of the actual distance cost or time relationship between nodal pairs. Indirect connections between nodes were determined by a matrix-powering procedure using Boolean rules. Thus, an L-matrix cell value could represent the shortest distance between \( i \) and \( j \). The row sum for a given \( i \) would indicate the total distance between \( i \) and all other nodes, over the shortest paths to each. The total of the row sums would indicate the total distance between each node in the network and all other nodes. As in the case of the D-matrix, the lower this figure, the more accessible the network.

These matrices were used to evaluate the relative merits of three different additions to the Ohio Interstate system, to compare the accessibility patterns associated with the highway and rail systems at regional and national scales, and to evaluate the accessibility of different cities in the initial development of the Amtrak system. In general, the emphasis was on the network impact on the accessibility of individual nodes to the rest of the system. In Chapter 13, we will place greater emphasis on the analysis of linkages and linkage systems such as hub-and-spoke networks.

The matrices discussed in this chapter are fundamentally descriptive in nature and are designed to provide more precise measurements of network structure and the comparative accessibility of nodes within them. In the next chapter, we will consider a normative or optimizing approach to transport analysis more thoroughly.